EXAMPLES AND APPLICATIONS OF INFINITE ITERATED FUNCTION SYSTEMS

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The aim of this work is the study of infinite conformal iterated function systems. More specifically, we investigate some properties of a limit set J associated to such system, its Hausdorff and packing measure and Hausdorff dimension. We provide necessary and sufficient conditions for such systems to be bi-Lipschitz equivalent. We use the concept of scaling functions to obtain some result about 1-dimensional systems.

We discuss particular examples of infinite iterated function systems derived from complex continued fraction expansions with restricted entries. Each system is obtained from an infinite number of contractions. We show that under certain conditions the limit sets of such systems possess zero Hausdorff measure and positive finite packing measure. We include an algorithm for an approximation of the Hausdorff dimension of limit sets. One numerical result is presented.

In this thesis we also explore the concept of positively recurrent function. We use iterated function systems to construct a natural, wide class of such functions that have strong ergodic properties.

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CHAPTER 1

INTRODUCTION

The purpose of this work is to collect together the results of Mariusz Urbański and myself presented and proven in [HU1], [HU2], and [HU3]. In all three papers we used the concept of infinite conformal iterated function system that was introduced and developed in [MU1]. Here we bring together three areas in which we use such systems to establish new results.

Chapter 1 contains some general definitions and notation which will be needed throughout this thesis. Basic ideas and some important results from the iterated function system and measure and dimension theory are presented.

Chapter 2 is devoted to the study of 1-dimensional systems. We provide necessary and sufficient conditions for such systems to be bi-Lipschitz equivalent. We extend to such systems the concept of scaling functions and pay special attention to the real-analytic systems.

In Chapter 3 we investigate special infinite iterated function systems derived from complex continued fraction expansions with restricted entries. We focus our attention on the corresponding limit set whose Hausdorff dimension will be denoted by h. Our primary goal is to determine whether the h-dimensional Hausdorff and packing measure of the limit set is positive and finite.

Additionally we provide an algorithm that shows how to approximate the Hausdorff dimensions of such systems. One numerical result improving the estimation from [MU1] is presented.

Finally, in the last two chapters we follow the concept of positively recurrent functions introduced and explored by Sarig in [Sa]. In Chapter 4 we use the concept of iterated function system to construct a natural wide class of positively recurrent functions and we show that they have stronger properties than the general functions considered in [Sa]. Our exposition closely follows the method of deriving the existence of invariant probability measure equivalent to the conformal measure presented in [HMU, MU1, and U]. In Chapter 5 we further investigate the σ-invariant measure $\tilde{\mu}_\phi$ introduced in Theorem 4.10. We show that this measure is the only equilibrium state for the potential ϕ.

1.1 Preliminaries

Specifically, let (X, ρ) be a compact metric space, and let I be a countable set with at least two elements. Define $S = \{\phi_i : X \to X \mid i \in I\}$, a collection of injective contractions from X to X for which there exists $0 < s < 1$ such that

$$\rho(\phi_i(x), \phi_i(y)) \leq s\rho(x, y),$$

for every $i \in I$ and for every pair of points $x, y \in X$. Thus, the system S is uniformly contractive. Any such collection is called an *iterated function system* (abbreviated as i.f.s.). We are particularly interested in the properties of the limit set J associated to the system $S = \{\phi_i : X \to X \mid i \in I\}$. We can define this set as the image of the coding space under a coding map as follows. Set $I^* = \bigcup_{m\geq 1} I^m$, the space of finite words, and, for $\omega \in I^m, m \geq 1$, define

$$\phi_\omega = \phi_{\omega_1} \circ \phi_{\omega_2} \circ \cdots \circ \phi_{\omega_m}.$$

If $\omega \in I^* \cup I^\infty$ and $m \geq 1$ does not exceed the length of ω, we denote by $\omega|_m$ the word $\omega_1\omega_2\ldots\omega_m$. Since given $\omega \in I^\infty$, the diameters of the compact sets $\phi_{\omega|_m}(X), m \geq 1$, converge to zero and since they form a descending family, the set

$$\bigcap_{m=1}^{\infty} \phi_{\omega|_m}(X)$$

is a singleton and therefore, denoting its only element by $\pi(\omega)$, defines the coding map $\pi : I^\infty \to X$. The main object of our interest will be the limit set

$$J = \pi(I^\infty) = \bigcup_{\omega\in I^\infty} \bigcap_{m=1}^{\infty} \phi_{\omega|_m}(X). \tag{1.1}$$

Observe that J satisfies the natural invariance equality, $J = \bigcup_{i\in I} \phi_i(J)$. Notice that if I is finite, then J is compact and this property fails for infinite systems.

An iterated function system, S, is said to satisfy the *Open Set Condition* (OSC) if there exists a nonempty open set $U \subset X$ such that $\phi_i(U) \subset U$ for all $i \in I$ and also $\phi_i(U) \cap \phi_j(U) = \emptyset$ for every pair $i, j \in I,\ i \neq j$. From now on assume that X is a subset of a d-dimensional Euclidean space. An iterated function system is said to be *conformal* if the following conditions are satisfied:

- X is compact and connected, $U = \mathrm{Int}_{\mathbb{R}^d}(X)$, $\phi_i(U) \subset U$, $\phi_i(U) \cap \phi_j(U) = \emptyset$ for $i \neq j$.
- There exist $\alpha, l > 0$ such that for every $x \in \partial X$ there exists an open cone,

$\mathrm{Con}(x, u_x, \alpha, l)$ with vertex x, direction vector u_x, central angle of Lebesgue measure α, and altitude l, that is contained in $\mathrm{Int}(X)$. This is the so-called cone property.

• There exists an open connected set $V \subset \mathbb{R}^d$ containing X such that every map ϕ_i can be extended to a $C^{1+\epsilon}$ diffeomorphism mapping V into V, and the extended map is conformal on V.

• There exists $K \geq 1$ such that $|\phi'_\omega(y)| \leq K|\phi'_\omega(x)|$ for every $\omega \in I^*$ and every pair of points $x, y \in V$. This is the Bounded Distortion Property (BDP).

In fact, sometimes we will need one condition more which can be considered as strenghtening of (BDP).

• There are two constants $L \geq 1$ and $\alpha > 0$ such that

$$||\phi'_i(y)| - |\phi'_i(x)|| \leq L||(\phi'_i)^{-1}||^{-1}|y - x|^\alpha.$$

for every $i \in I$ and every pair of points $x, y \in V$.

1.2 Hausdorff Dimension, Hausdorff and Packing Measures

A finite or countable collection of subsets $\{U_i\}$ of $\mathbb{R}^n$ is called a δ-*cover* of a Borel set $E \subset \mathbb{R}^n$ if $|U_i| = \mathrm{diam}(U_i) \leq \delta$ for all i and $E \subset \bigcup_{i=1}^\infty U_i$.

Let E be a subset of $\mathbb{R}^n$ and $s \geq 0$. For all $\delta > 0$ we define

$$\mathcal{H}^s_\delta(E) = \inf\left\{\sum_{i=1}^\infty |U_i|^s : \ \{U_i\} \text{ is a } \delta - \text{cover of } E\right\}. \tag{1.2}$$

As δ decreases, the class of δ-covers of E is reduced, so its infimum increases and approaches a limit as $\delta \searrow 0$. Thus we define

$$\mathcal{H}^s(E) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(E). \tag{1.3}$$

This limit exists, perhaps as 0 or ∞, for all $E \subset \mathbb{R}^n$.

We term $\mathcal{H}^s(E)$ the *s-dimensional Hausdorff measure* of E.

It may be shown that $\mathcal{H}^s(E)$ is a Borel regular measure on $\mathbb{R}^n$ (see for example [Ma]). It is also easy to show from (1.2) and (1.3) that for all sets $E \subset \mathbb{R}^n$ there is a number $\dim_H E$, called the *Hausdorff dimension* of E, such that $\mathcal{H}^s(E) = \infty$ if $s < \dim_H(E)$ and $\mathcal{H}^s(E) = 0$ if $s > \dim_H(E)$. Thus

$$\dim_H(E) = \inf\{s : \ \mathcal{H}^s(E) = 0\} = \sup\{s : \ \mathcal{H}^s(E) = \infty\},$$

so that the Hausdorff dimension of a set E may be thought of as the number s at which $\mathcal{H}^s(E)$ jumps from ∞ to 0. When $s = \dim_H(E)$ the measure $\mathcal{H}^s(E)$ can be zero or infinite, but in the nicest situation it is positive and finite, in which case E is called an *s-set*.

We define a *δ-packing* of $E \subset \mathbb{R}^n$ to be a finite or countable collection of disjoint (in $\mathbb{R}^n$) balls $\{B_i\}$ of radii at most δ and with centres in E. For $\delta > 0$ we define

$$\mathcal{P}^{s*}_\delta(E) = \sup\left\{\sum_{i=1}^{\infty} |B_i|^s : \ \{B_i\} \text{ is a } \delta - \text{packing of } E\right\}.$$

Clearly, if $0 < \delta_1 < \delta_2$, then $\mathcal{P}^{s*}_{\delta_1}(E) \leq \mathcal{P}^{s*}_{\delta_2}(E)$, so as $\delta \searrow 0$ we may take the limit

$$\mathcal{P}^{s*}(E) = \lim_{\delta \to 0} \mathcal{P}^{s*}_\delta(E).$$

Unfortunately $\mathcal{P}^{s*}$ is not a measure (or even outer measure). To overcome this difficulty one defines

$$\mathcal{P}^s(E) = \inf \left\{ \sum_{i=1}^{\infty} \mathcal{P}^{s*}(E_i) : \; E \subset \bigcup_{i=1}^{\infty} E_i \right\}$$

which is a Borel measure on $\mathbb{R}^n$, called the *s-dimensional packing measure* of E.

It may be shown (see [Ma]) that $\mathcal{H}^s(A) \leq \mathcal{P}^s(A)$ for all Borel sets A. It can also be proven (see Lemma 4.2 and Lemma 4.3 of [MU1]) that for a conformal iterated function system, if $h = \dim_H(J)$ then $\mathcal{H}^h(J) < \infty$ and $\mathcal{P}^h(J) > 0$. To prove $\mathcal{P}^h(J) > 0$ authors assume that the system preserves so called strong open set condition, but this assumption can be released due to new results of Mauldin, Solomyak, and Urbański from [MSU].

1.3 Topological Pressure and Conformal Measures

Let $X(\infty)$ be the set of limit points of all sequences $x_i \in \phi_i(X),\ i \in I'$, where I' ranges over all infinite subsets of I. $X(\infty)$ can be called an "asymptotic boundary" and the behavior of the system at this set affects the properties of the limit set J.

The topological pressure function, P, for iterated function systems is defined as follows. For every $t \geq 0$ consider the series

$$\psi_1(t) = \sum_{i \in I} ||\phi_i'||^t,$$

and more generally define for every integer $n \geq 1$

$$\psi_n(t) = \sum_{\omega \in I^n} ||\phi_\omega'||^t.$$

Now set

$$P(t) = \lim_{n\to\infty} \frac{1}{n} \log \psi_n(t). \tag{1.4}$$

Detailed properties of this pressure function can be found in [MU1]. In [MU3] its definition is extended to case of parabolic iterated function systems and in [HMU, HU2, U], the topological pressure of systems of Hölder continuous functions is defined and explored. This last concept also generalizes formula (1.4). As shown in [MU1], there are two disjoint classes of conformal iterated function systems, regular and irregular. A system is *regular* if there exists $t \geq 0$ such that $P(t) = 0$. Otherwise the system is *irregular*. The following property demonstrating the geometrical significance of topological pressure holds (see [MU1, Theorem 3.15]).

Theorem 1.1 $\dim_H(J) = \sup\{\dim_H(J_F) : \ F \subset I \text{ finite}\} = \inf\{\mathrm{t} \geq 0 : \ \mathrm{P(t)} \leq 0\}$. *If a system is regular and* $P(t) = 0$ *then* $t = \dim_H(J)$.

A Borel probability measure m is said to be t-*conformal* if $m(J) = 1$ and for every Borel set A and every $i \in I$,

$$m(\phi_i(A)) = \int_A |\phi_i'|^t \, dm \tag{1.5}$$

and

$$m(\phi_i(X) \cap \phi_j(X)) = 0, \tag{1.6}$$

for every pair $i, j \in I, \ i \neq j$. Lemma 3.13 in [MU1] shows that the conformal iterated function system is regular if and only if there exists a t-conformal measure (t is such that $P(t) = 0$), and then $t = \dim_H(J)$.

CHAPTER 2

RIGIDITY OF ONE-DIMENSIONAL SYSTEMS

2.1 General Systems

Throughout this chapter we will assume that the index set is the set of natural numbers, that is $I = \mathbb{N}$.

We call two iterated function systems $\{f_i : X \to X, i \in \mathbb{N}\}$ and $\{g_i : Y \to Y, i \in \mathbb{N}\}$ *topologically conjugate* if and only if there exists a homeomorphism $h : J_F \to J_G$ such that $h \circ f_i = g_i \circ h$ for all $i \in \mathbb{N}$. Then by induction we easily get that $h \circ f_\omega = g_\omega \circ h$ for every finite word ω.

The main result of this section is the following assertion.

Theorem 2.1 *Suppose that $F = \{f_i : X \to X, i \in \mathbb{N}\}$ and $G = \{g_i : Y \to Y, i \in \mathbb{N}\}$ are two topologically conjugate conformal iterated function systems. Then the following 4 conditions are equivalent.*

(1) $\exists S \geq 1 \ \forall \omega \in \mathbb{N}^*$

$$S^{-1} \leq \frac{\operatorname{diam}(g_\omega(Y))}{\operatorname{diam}(f_\omega(X))} \leq S.$$

(2) $|g'_\omega(y_\omega)| = |f'_\omega(x_\omega)|$ *for all* $\omega \in \mathbb{N}^*$*, where* x_ω *and* y_ω *are the only fixed points of* $f_\omega : X \to X$ *and* $g_\omega : Y \to Y$ *respectively.*

(3) $\exists E \geq 1\ \forall \omega \in \mathbb{N}^*$

$$E^{-1} \leq \frac{||g'_\omega||}{||f'_\omega||} \leq E.$$

(4) For every finite subset T of $\mathbb{N}$, $\dim_H(J_{G,T}) = \dim_H(J_{F,T})$ and the conformal measures $m_{G,T}$ and $m_{F,T} \circ h^{-1}$ are equivalent.

Suppose additionally that both systems F and G are regular. Then the following condition is also equivalent with the four conditions above.

(5) $\dim_H(J_G) = \dim_H(J_F)$ *and the conformal measures m_G and $m_F \circ h^{-1}$ are equivalent.*

Proof Let us first demonstrate that conditions (2) and (3) are equivalent. Indeed, suppose that (2) is satisfied and let K_F and K_G denote the distortion constants of the systems F and G respectively. Then for all $\omega \in \mathbb{N}^*$, $||g'_\omega|| \leq K_G|g'_\omega(y_\omega)| = K_G|f'_\omega(x_\omega)| \leq K_G||f'_\omega||$ and similarly $||f'_\omega|| \leq K_F||g'_\omega||$. So suppose that (3) holds and (2) fails, that is that there exists $\omega \in \mathbb{N}^*$ such that $|g'_\omega(y_\omega)| \neq |f'_\omega(x_\omega)|$. Without loosing generality we may assume that $|g'_\omega(y_\omega)| < |f'_\omega(x_\omega)|$. For every $n \geq 1$ let ω^n be the concatenation of n words ω. Then $g_{\omega^n}(y_\omega) = g^n_\omega(y_\omega) = y_\omega$ and similarly $f_{\omega^n}(x_\omega) = x_\omega$. So, $x_{\omega^n} = x_\omega = \pi_F(\omega^\infty)$ and $y_{\omega^n} = y_\omega = \pi_G(\omega^\infty)$. Moreover $|g'_{\omega^n}(y_\omega)| = |g'_\omega(y_\omega)|^n$ and $|f'_{\omega^n}(x_\omega)| = |f'_\omega(x_\omega)|^n$. Hence

$$\lim_{n\to\infty} \frac{|g'_{\omega^n}(y_\omega)|}{|f'_{\omega^n}(x_\omega)|} = 0.$$

On the other hand, by (3) and the Bounded Distortion Property

$$\frac{|g'_{\omega^n}(y_\omega)|}{|f'_{\omega^n}(x_\omega)|} \geq \frac{K_G^{-1}||g'_{\omega^n}||}{||f'_{\omega^n}||} \geq E^{-1}K_G^{-1}$$

for all $n \geq 1$. This contradiction finishes the proof of equivalence of conditions (2) and (3). Since the equivalence of (1) and (3) is immediate, the proof of the equivalence of conditions (1)–(3) is finished.

We shall now prove that (3) $\Rightarrow$ (5). Indeed, it follows from (3) that $E^{-1}\psi_{G,n}(t) \leq \psi_{F,n}(t) \leq E\psi_{G,n}(t)$ for all $t \geq 0$ and all $n \geq 1$. Hence $P_G(t) = P_F(t)$ and therefore by Theorem 1.1, $\dim_H(J_G) = \dim_H(J_F)$. Denote this common value by h. Although we keep the same symbol for the homeomprphism establishing conjugacy between the systems F and G, it will never cause misunderstandings. Suppose now that both systems are regular (in fact assuming (3) regularity of one of these systems implies regularity of the other). Then for every $\omega \in \mathbb{N}^*$

$$(K_F E)^{-h} \leq \frac{K_F^{-h}||f'_\omega||^h}{||g'_\omega||^h} \leq \frac{m_F(f_\omega(J_F))}{m_G(g_\omega(J_G))} \leq \frac{||f'_\omega||^h}{K_G^{-h}||g'_\omega||^h} \leq (EK_G)^h.$$

So, the measures m_G and $m_F \circ h^{-1}$ are equivalent, and even more

$$(K_F E)^{-h} \leq \frac{dm_F \circ h^{-1}}{dm_G} \leq (EK_G)^h.$$

Let us show now that (5) $\Rightarrow$ (3). Indeed, if (5) is satisfied then the measure $\mu_F \circ h^{-1}$ is equivalent with μ_G. Since additionally $\mu_F \circ h^{-1}$ and μ_G are both ergodic with respect to the shift map on $\mathbb{N}^\infty$ (see Theorem 3.8 of [MU1]), they are equal. Hence, using the equality $\dim_H(J_F) = \dim_H(J_G) = h$, we get

$$\begin{aligned}
||g'_\omega||^h &\asymp \int |g'_\omega|^h \, dm_G = m_G(g_\omega(J_G)) \asymp \mu_G(\pi^{-1}(g_\omega(J_G))) \\
&= \mu_F \circ h^{-1}(\pi^{-1}(g(J_G))) = \mu_F(\pi^{-1}(f_\omega(J_F))) \asymp m_F(f_\omega(J_F)) \\
&= \int |f'_\omega|^h \, dm_F \asymp ||f'_\omega||^h,
\end{aligned}$$

and raising the first and the last term of this sequence of comparabilities to the power $1/h$, we finish the proof of the implication (5) $\Rightarrow$ (3).

The equivalence of (4) and conditions (1)–(3) is now a relatively simple corollary. Indeed, to prove that (3) implies (4) fix a finite subset T of $\mathbb{N}$. By (3) $E^{-1} \leq ||f'_\omega||/||g'_\omega|| \leq E$ for all $\omega \in T^*$, and as every finite system is regular, the equivalence of measures $m_{G,T}$ and $m_{F,T} \circ h^{-1}$ follows from the equivalence of conditions (3) and (5) applied to the systems $\{f_i : i \in T\}$ and $\{g_i : i \in T\}$. If in turn (4) holds and $\omega \in \mathbb{N}^*$, then $\omega \in T^*$, where T is the (finite) set of letters making up the word ω and the measures $m_{G,T}$ and $m_{F,T} \circ h^{-1}$ are equivalent. Hence, by the equivalence of (2) and (5) applied to the systems $\{f_i : i \in T\}$ and $\{g_i : i \in T\}$ we conclude that $|g'_\omega(y_\omega)| = |f'_\omega(x_\omega)|$. Thus (2) follows and the proof of Theorem 2.1 is finished. ■

We say that a conformal system $\{\phi_i : X \to X : i \in \mathbb{N}\}$ is *of bounded geometry* if and only if there exists $C \geq 1$ such that for all $i, j \in \mathbb{N}$, $i \neq j$

$$\max\{\mathrm{diam}(\phi_i(X)), \mathrm{diam}(\phi_j(X))\} \leq C\mathrm{dist}(\phi_i(X), \phi_j(X)).$$

The next theorem provides a necessary and sufficient condition for two systems of bounded geometry to be bi-Lipschitz equivalent.

Theorem 2.2 *If both systems $\{f_i : X \to X : i \in \mathbb{N}\}$ and $\{g_i : Y \to Y : i \in \mathbb{N}\}$ are of bounded geometry, then the topological conjugacy $h : J_f \to J_g$ is bi-Lipschitz continuous if and only if the following two conditions are satisfied.*

(a) $Q^{-1} \leq \frac{\mathrm{diam}(f_\omega(X))}{\mathrm{diam}(g_\omega(Y))} \leq Q$ for some $Q \geq 1$ and all $\omega \in \mathbb{N}^*$.

(b) $D^{-1} \leq \frac{\mathrm{dist}(g_i(Y),g_j(Y))}{\mathrm{dist}(f_i(X),f_j(X))} \leq D$ *for some* $D \geq 1$ *and all* $i,j \in \mathbb{N},\ i \neq j$.

Proof First notice that, due to the boundedness of geometry of F and G, (a) and (b) remain true, with modified constants Q and D if necessary, if X is replaced by J_F and Y is replaced by J_G respectively. Suppose now that $x \in f_i(J_F)$ and $y \in f_j(J_F)$ with $i \neq j$. Then

$$\begin{aligned}
|h(y)-h(x)| &\leq \mathrm{diam}(g_i(J_G)) + \mathrm{dist}(g_i(J_G), g_j(J_G)) + \mathrm{diam}(g_j(J_g)) \\
&\leq Q\mathrm{diam}(f_i(J_F)) + D\mathrm{dist}(f_i(J_F), f_j(J_F)) + Q\mathrm{diam}(f_j(J_F)) \\
&\leq 2QC\mathrm{dist}(f_i(J_F), f_j(J_F)) + D\mathrm{dist}(f_i(J_F), f_j(J_F)) \\
&\leq (2QC+D)\mathrm{dist}(f_i(J_F), f_j(J_F)) \\
&\leq (2QC+D)|y-x|
\end{aligned}$$

Suppose in turn that $x \neq y$ both belong to the same element $f_k(J_F)$. Then there exist $\omega \in \mathbb{N}^*$ ($|\omega| \geq 1$) and $i \neq j \in \mathbb{N}$ such that $x, y \in f_\omega(J_F)$, $x \in f_{\omega i}(J_F)$ and $y \in f_{\omega j}(J_F)$. From what has been proved so far we know that $|g_\omega^{-1}(h(y)) - g_\omega^{-1}(h(x))| \leq (2QC+D)|f_\omega^{-1}(y) - f_\omega^{-1}(x)|$. Since $|y-x| \asymp ||f'_\omega|| \cdot |f_\omega^{-1}(y) - f_\omega^{-1}(x)|$ and $|h(y)-h(x)| \asymp ||g'_\omega|| \cdot |g_\omega^{-1}(h(y)) - |g_\omega^{-1}(h(x))|$, we get

$$|h(y)-h(x)| \leq \mathrm{const}\frac{||g'_\omega||}{||f'_\omega||}|y-x| \asymp |y-x|,$$

where the comparability sign we have written due to (a) and the equivalence of conditions (1) and (3) of Theorem 2.1. In the same way we show that h^{-1} is Lipschitz continuous which completes the proof of the first part of our theorem.

So suppose now that h is bi-Lipschitz continuous. We shall show that conditions (a) and (b) are satisfied. Indeed, to prove (a) suppose that a and b in $f_\omega(J_F)$ are taken so that $|h(a)-h(b)| \geq \frac{1}{2}\text{diam}(g_\omega(J_G))$. Then

$$\text{diam}(g_\omega(J_G)) \leq 2|h(a)-h(b)| \leq 2L|a-b| \leq 2L\text{diam}(f_\omega(J_F)),$$

where L is a Lipschitz constant of h and h^{-1}. In the same way it can be shown that $\text{diam}(f_\omega(J_F)) \leq 2L\text{diam}(g_\omega(J_G))$ which completes the proof of property (a). In order to prove the right-hand side of property (b) we proceed as follows. Fix $i, j \in \mathbb{N}$, $i \neq j$ and $a \neq b \in J_F$. Then

$$\begin{aligned}
\text{dist}(g_i(Y), g_j(Y)) &\leq \text{dist}(g_i(J_G), g_j(J_G)) \leq |g_i(h(a)) - g_j(h(b))| \\
&= |h(f_i(a)) - h(f_j(b))| \leq L|f_i(a) - f_j(b)| \\
&\leq L(\text{diam}(f_i(X)) + \text{dist}(f_i(X), f_j(X)) + \text{diam}(f_j(X))) \\
&\leq L(2C+1)\text{dist}(f_i(X), f_j(X)),
\end{aligned}$$

where the last inequality we wrote due to boundedness of geometry of the system $\{f_i : i \in \mathbb{N}\}$. The proof is finished. ■

Remark 2.3 *Notice that in Theorem 2.1 and Theorem 2.2 we do not assume that the phase space X is one-dimensional. We only need to know that the maps f_i and g_i are conformal and the assumption (2.7) of [MU1] is satisfied.*

From now on we assume that the space X is one-dimensional.

Remark 2.4 *Suppose now that the maps $i \mapsto \phi_i(X)$ are monotone, that is suppose that for all i and j, $i < j$ implies $\phi_i(X) < \phi_j(X)$. We claim that then the bounded geometry of the system is equivalent with the following weaker condition*

$$\max\{\mathrm{diam}(\phi_{\mathrm{i}}(\mathrm{X})), \mathrm{diam}(\phi_{\mathrm{i+1}}(\mathrm{X}))\} \leq \mathrm{Cdist}(\phi_{\mathrm{i}}(\mathrm{X}), \phi_{\mathrm{i+1}}(\mathrm{X})).$$

Indeed, if $i < j$, then

$$\begin{aligned}
\max\{\mathrm{diam}(\phi_i(X), \mathrm{diam}(\phi_j(X)\} &\leq \max_{i\leq k\leq j-1}\{\max\{\mathrm{diam}(\phi_k(X)), \mathrm{diam}(\phi_{k+1}(X))\}\} \\
&\leq \max_{i\leq k\leq j-1}\{C\mathrm{dist}(\phi_k(X), \phi_{k+1}(X))\} \\
&\leq C\mathrm{dist}(\phi_i(X), \phi_j(X)),
\end{aligned}$$

where writing the last inequality we used the monotonicity of the map $i \mapsto \phi_i(X)$. The opposite implication is obvious.

Remark 2.5 *If both maps $i \mapsto f_i(X)$ and $i \mapsto g_i(X)$ are monotone, then condition (b) from Theorem 2.2 can be replaced by the following.*

(c) $C^{-1} \leq \frac{\mathrm{dist}(g_k(Y), g_{k+1}(Y))}{\mathrm{dist}(f_k(X), f_{k+1}(X))} \leq C$

for some constant $C \geq 1$ and all $k \in \mathbb{N}$. Indeed, assuming (c) this follows from the following computation.

$$\begin{aligned}
\mathrm{dist}(g_i(Y), g_j(Y)) &= \sum_{k=i}^{j-1}\mathrm{dist}(g_k(Y), g_{k+1}(Y)) + \sum_{k=i+1}^{j-1}\mathrm{diam}(g_k(X)) \\
&\leq \sum_{k=i}^{j-1}C\mathrm{dist}(f_k(X), f_{k+1}(X)) + Q\sum_{k=i+1}^{j-1}\mathrm{diam}(f_k(X)) \\
&\leq \max\{C, Q\}\Big(\sum_{k=i}^{j-1}\mathrm{dist}(f_k(X), f_{k+1}(X))
\end{aligned}$$

$$+\sum_{k=i+1}^{j-1} \operatorname{diam}(f_k(X))\Big)$$
$$= \max\{C,Q\} dist(f_i(X), f_j(X))$$

2.2 Real-analytic Systems

We call a 1-dimensional system $\Phi = \{\phi_i : X \to X, i \in \mathbb{N}\}$ *real-analytic* if and only if there exists a topological disk D such that all the maps ϕ_i extend in a conformal (so 1-to-1) fashion to D into D.

Suppose that Φ is regular. Let m be the conformal measure associated to the system Φ and let μ be the only probability invariant measure equivalent with m (see [MU1], Theorem 3.8, where this measure was denoted by μ^*). We call the system Φ non-linear (comp. [S1]) if and only if at least one of the Jacobians $J_{\phi_i} = \frac{d\mu\circ\phi_i}{d\mu}$ is not constant. We shall prove the following theorem which is stronger than both Theorem 2.1 and Theorem 2.2.

Theorem 2.6 *If both systems $\{f_i : X \to X : i \in \mathbb{N}\}$ and $\{g_i : Y \to Y : i \in \mathbb{N}\}$ are real-analytic and non-linear, then the following conditions are equivalent.*

(a) The conjugacy between the systems $\{f_i : X \to X : i \in \mathbb{N}\}$ and $\{g_i : Y \to Y : i \in \mathbb{N}\}$ extends in a real-analytic fashion to the convex hull of J_F.

(b) The conjugacy between the systems $\{f_i : X \to X : i \in \mathbb{N}\}$ and $\{g_i : Y \to Y : i \in \mathbb{N}\}$ is bi-Lipschitz continuous.

(c) $|g'_\omega(y_\omega)| = |f'_\omega(x_\omega)|$ for all $\omega \in \mathbb{N}^*$, where x_ω and y_ω are the only fixed points of

$f_\omega : X \to X$ and $g_\omega : Y \to Y$ respectively.

(d) $\exists S \geq 1 \; \forall \omega \in \mathbb{N}^*$

$$S^{-1} \leq \frac{\mathrm{diam}(g_\omega(Y))}{\mathrm{diam}(f_\omega(X))} \leq S.$$

(e) $\exists E \geq 1 \; \forall \omega \in \mathbb{N}^*$

$$E^{-1} \leq \frac{||g'_\omega||}{||f'_\omega||} \leq E.$$

(f) $\dim_H(J_G) = \dim_H(J_F)$ *and the measures m_G and $m_F \circ h^{-1}$ are equivalent.*

(g) The measures m_G and $m_F \circ h^{-1}$ are equivalent.

Proof The implication $(a) \Rightarrow (b)$ is obvious. That $(b) \Rightarrow (c)$ results from the fact that (b) implies condition (1) of Theorem 2.1 which in view of this theorem is equivalent with condition (2) of Theorem 2.1 which finally is the same as condition (c) of Theorem 2.6. The implications $(c) \Rightarrow (d) \Rightarrow (e) \Rightarrow (f)$ have been proven in Theorem 2.1. The implication $(f) \Rightarrow (g)$ is again obvious. We are left to prove that $(g) \Rightarrow (a)$. As the first step we shall show that if a regular system $\{\phi_i : i \in \mathbb{N}\}$ is real-analytic, then the Jacobians J_{ϕ_ω} of all the maps ϕ_ω, $\omega \in \mathbb{N}^*$ with respect to the invariant measure μ are also real-analytic. Since $\frac{d(m \circ \phi_\omega)}{dm} = |\phi'_\omega|^h$ and since $\phi'_\omega|_X$ is a real valued, either positive or negative, real-analytic function, the function $|\phi'_\omega|^h$ is also real analytic. Consequently, to check that

$$\frac{d\mu \circ \phi_\omega}{d\mu} = \frac{d\mu \circ \phi_\omega}{dm \circ \phi_\omega} \cdot \frac{dm \circ \phi_\omega}{dm} \cdot \frac{dm}{d\mu} = \frac{d\mu}{dm} \circ \phi_\omega \cdot \frac{dm \circ \phi_\omega}{dm} \cdot \frac{dm}{d\mu}$$

is real-analytic it suffices to check that $\frac{d\mu}{dm}$ is real-analytic. Let $D \subset \overline{\mathbb{C}}$ be the open topological disk claimed in the definition of real analytic systems. Since for each

$\omega \in \mathbb{N}^*$, $|\phi'_\omega||_X = \pm\phi'_\omega|_X$, all the derivatives extend (complex) analytically to the corresponding maps $\nu(\omega)\phi'_\omega$, where $\nu(\omega) \in \{1, -1\}$. Given $n \geq 1$ consider the series of (complex) analytic functions $\mathcal{L}^n(1) = \sum_{|\omega|=n}(\nu(\omega)\phi'_\omega)^h$, where $(\nu(\omega)\phi'_\omega)^h$ are well-defined since D is simply connected. Fix $x_0 \in X$. By the Koebe Distortion Theorem and (3.3) of [MU1] we can write for all $n \geq 1$ and all $x \in D$

$$\left|\sum_{|\omega|=n}(\nu(\omega)\phi'_\omega)^h\right| \leq \sum_{|\omega|=n}|\phi'_\omega(x)|^h \leq K^h \sum_{|\omega|=n}|\phi'_\omega(x_0)|^h = K^h\mathcal{L}^n(1)(x_0) \leq K^{2h}.$$

Hence, the maps $\mathcal{L}^n(1) : D \to \mathbb{C}$ form a normal family in the sense of Montel. Since for m-a.e. point x, $\mathcal{L}^n(1)|_X$ converges to $\rho(x) = \frac{d\mu}{dm}(x)$, we conclude more, that $\mathcal{L}^n(1)|_D$ converges to an analytic extension of ρ on D. We will keep the same notation ρ for this extension. So, we have proven that all the Jacobians $J_{\phi_\omega} = \frac{d\mu\circ\phi_\omega}{d\mu}$ are real-analytic, and in fact, extend analytically onto D. Now suppose that condition (g) of Theorem 2.6 is satisfied. Then $\mu_F = \mu_G \circ h$ meaning that $J_h = \frac{d\mu_G \circ h}{d\mu_F} = 1$. Since $h \circ f_\omega = g_\omega \circ h$, the chain rule implies that $J_h \circ f_\omega \cdot J_{f_\omega} = J_{g_\omega} \circ h \cdot J_h$ and consequently $J_{f_\omega} = J_{g_\omega} \circ h$.

Let now g_i be the contraction produced by non-linearity. Notice then that J_{g_i} has only finitely many extremal points, since otherwise the equation $J'_{g_i} = 0$ would have an accumulation point in Y which in turn would imply that J_{g_i} would be constant on Y, contrary to the choice of g_i. Hence $J_{g_i}^{-1} \circ J_{f_i}$ is well-defined and 1-to-1 on an open set $V \subset X$, and $h = J_{g_i}^{-1} \circ J_{f_i}$ on $V \cap J_F$. Consider now $\omega \in \mathbb{N}^*$ such that $f_\omega(X) \subset V$. Denote by C_F the convex hull of J_F. Then the map $g_\omega^{-1} \circ (J_{g_i}^{-1} \circ J_{f_i}) \circ f_\omega : C_F \mapsto Y$ is well defined, extends h, and is real-analytic. ■

We should mention that the last result has been extended in [MPU] to a 2-dimensional case and then the analogical result has been obtained in [PU2] in case $d \geq 3$. Namely if we additionally assume that both systems $\{f_i : X \to X : i \in \mathbb{N}\}$ and $\{g_i : Y \to Y : i \in \mathbb{N}\}$ are conformal, regular, and satisfy Open Set Condition then the claims (b)–(g) from Theorem 2.6 are equivalent to condition that the conjugacy between the systems extends in a conformal (or real-analytic) fashion to an open neighbourhood of X.

2.3 Scaling Functions

Let us now refer to the stronger version of (BDP) condition. As a byproduct of the demonstration that $(b) \Rightarrow (c)$ (p.112 of [MU1]) we have shown that for all $\omega \in \mathbb{N}^*$, say $\omega \in \mathbb{N}^n$, and all $x, y \in X$

$$|\log|\phi'_\omega(y)| - \log|\phi'_\omega(x)|| \leq \sum_{j=1}^{n} ||(\phi'_{\omega_j})^{-1}|| \cdot (|\phi'_{\omega_j}(y_{n-j})| - |\phi'_{\omega_j}(x_{n-j})|),$$

where $z_k = \phi_{\omega_{n-k+1}} \circ \cdots \circ \phi_{\omega_n}(z)$. In view of stronger version of (BDP) this estimate continuous as follows.

$$\begin{aligned} |\log|\phi'_\omega(y)| - \log|\phi'_\omega(x)|| &\leq \sum_{j=1}^{n} L|y_{n-j} - x_{n-j}|^\alpha \\ &\leq \sum_{j=0}^{n-1} L s^{j\alpha}|y - x|^\alpha \\ &= \frac{L}{1 - s^\alpha}|y - x|^\alpha \end{aligned} \tag{2.1}$$

or equivalently

$$\exp\left(\frac{-L}{1 - s^\alpha}|y - x|^\alpha\right) \leq \frac{|\phi'_\omega(y)|}{|\phi'_\omega(x)|} \leq \exp\left(\frac{L}{1 - s^\alpha}|y - x|^\alpha\right) \tag{2.2}$$

Now, since for every $t \geq 0$ sufficiently small $|e^t - 1| \leq 2t$, we get

$$\begin{aligned} ||\phi'_\omega(y)| - |\phi'_\omega(x)|| &= \left| \frac{|\phi'_\omega(y)|}{|\phi'_\omega(x)|} - 1 \right| |\phi'_\omega(x)| \\ &\leq \frac{2L}{1-s^\alpha}|y-x|^\alpha |\phi'_\omega(x)| \leq \frac{2Ls^{|\omega|}}{1-s^\alpha}|y-x|^\alpha \end{aligned} \tag{2.3}$$

In order to define scaling functions we will need the following basic lemma.

Lemma 2.7 *If $\{\phi_n : X \to X : n \geq 1\}$ is a one-dimensional conformal iterated function system satisfying stronger (BDP) condition, then for every closed subinterval K of X and $\omega \in \mathbb{N}^\infty$*

$$\lim_{n\to\infty} \frac{|\phi_{\omega_n\omega_{n-1}\ldots\omega_0}(K)|}{|\phi_{\omega_n\omega_{n-1}\ldots\omega_0}(X)|} := S(\omega, K)$$

exists and the convergence is uniform with respect to $K, n,$ and ω.

Proof We shall show that the above sequence satisfies an appropriate Cauchy condition. So, fix $k < n$. We then have

$$\frac{\frac{|\phi_{\omega_n\ldots\omega_k\ldots\omega_0}(K)|}{|\phi_{\omega_n\ldots\omega_k\ldots\omega_0}(X)|}}{\frac{|\phi_{\omega_k\ldots\omega_0}(K)|}{|\phi_{\omega_k\ldots\omega_0}(X)|}} = \frac{\frac{|\phi_{\omega_n\ldots\omega_{k+1}}(\phi_{\omega_k\ldots\omega_0}(K))|}{|\phi_{\omega_k\ldots\omega_0}(K)|}}{\frac{|\phi_{\omega_n\ldots\omega_{k+1}}(\phi_{\omega_k\ldots\omega_0}(X))|}{|\phi_{\omega_k\ldots\omega_0}(X)|}} = \frac{|\phi'_{\omega_n\ldots\omega_{k+1}}(x_n)|}{|\phi'_{\omega_n\ldots\omega_{k+1}}(y_n)|} \tag{2.4}$$

for some $x_n \in \phi_{\omega_k\ldots\omega_0}(K)$ and $y_n \in \phi_{\omega_k\ldots\omega_0}(X)$, where the last equality sign we wrote due to the Mean Value Theorem. Denote now $|\phi_{\omega_j\ldots\omega_0}(K)|/|\phi_{\omega_j\ldots\omega_0}(X)|$ by a_j. In view of (2.4) and (2.1) we get

$$|\log a_n - \log a_k| \leq \frac{L}{1-s^\alpha}|x_n - y_n|^\alpha \leq \frac{L}{1-s^\alpha}|\phi_{\omega_k\ldots\omega_0}(X)|^\alpha \leq \frac{L}{1-s^\alpha}s^{k\alpha}.$$

Thus the sequence $\{\log a_n\}_{n=1}^\infty$ is a Cauchy sequence, and consequently $\{a_n\}_{n=1}^\infty$ itself is also a Cauchy sequence. The proof is finished. ■

Let $\tilde{\mathbb{N}}^\infty$ denote the set of infinite sequences of the form $\dots\omega_n\omega_{n-1}\dots\omega_1\omega_0$ and let $\tilde{\mathbb{N}}^*$ denote the set of all finite words of the form $\omega_n\omega_{n-1}\dots\omega_1\omega_0$. Lemma 2.7 allows us to introduce the scaling function (comp. also [S2] and [PT]). In this section we will explore this notion. The weaker scaling function S^w is defined on the space $\tilde{\mathbb{N}}^\infty \times \mathbb{N}$, takes values in $(0,1)$, and is given by the formula

$$S^w(\{\omega_n\}_{n=0}^\infty, i) = \lim_{n\to\infty} \frac{|\phi_{\omega_n\omega_{n-1}\dots\omega_0}(\phi_i(X))|}{|\phi_{\omega_n\omega_{n-1}\dots\omega_0}(X)|},$$

where the limit exists due to Lemma 2.7.

The stronger scaling function S^s is defined similarly but on the larger space $\tilde{\mathbb{N}} \times (\mathbb{N} \cup \mathcal{C})$, where $\mathcal{C}$ denotes the set of all connected components of $X \setminus \bigcup_{i=1}^\infty \phi_i(X)$. Frequently, given $\omega \in \mathbb{N}^*$ we will consider the function $S^s(\omega) : (\mathbb{N} \cup \mathcal{C}) \to (0,1)$ given by the formula $S^s(\omega)(Z) = S^s(\omega, Z)$, and similarly we define the function $S^w(\omega)$. The following theorem is an immediate consequence of Lemma 2.7.

Theorem 2.8 *Both scaling functions $S^w : \tilde{\mathbb{N}}^\infty \times \mathbb{N} \to (0,1)$ and $S^s : \tilde{\mathbb{N}} \times (\mathbb{N} \cup \mathcal{C}) \to (0,1)$ are continuous.*

Our last theorem reads as follows.

Theorem 2.9 *If the topological conjugacy $h : J_F \to J_G$ extends in a diffeomorphic fashion onto X, then J_F and J_G have the same strong scaling functions. If conversely, two topologically conjugate 1-dimensional iterated function systems F and G of bounded geometry have the same weak scaling functions and condition (b) of Theorem 2.2 is satisfied, then the topological conjugacy is bi-Lipschitz continuous.*

Proof Let us first prove the first part of this theorem. Indeed, let us keep the same notation h for its diffeomorphic extension to X and let D be an arbitrary closed subinterval of X. For $\omega \in \tilde{\mathbb{N}}^\infty$ we can write

$$\frac{S(\omega, D)}{S(\omega, h(D))} = \lim_{n\to\infty} \frac{\frac{|f_{\omega_n\ldots\omega_0}(D)|}{|f_{\omega_n\ldots\omega_0}(X)|}}{\frac{|g_{\omega_n\ldots\omega_0}(h(D))|}{|g_{\omega_n\ldots\omega_0}(Y)|}} = \lim_{n\to\infty} \frac{\frac{|f_{\omega_n\ldots\omega_0}(D)|}{|g_{\omega_n\ldots\omega_0}(h(D))|}}{\frac{|f_{\omega_n\ldots\omega_0}(X)|}{|g_{\omega_n\ldots\omega_0}(Y)|}}.$$

Now, by the Mean Value Theorem there exist a_n and b_n respectively in $f_{\omega_n\ldots\omega_0}(D)$ and in $f_{\omega_n\ldots\omega_0}(X)$ such that

$$\frac{S(\omega, D)}{S(\omega, h(D))} = \lim_{n\to\infty} \frac{\frac{|f_{\omega_n\ldots\omega_0}(D)|}{|h(f_{\omega_n\ldots\omega_0}(D))|}}{\frac{|f_{\omega_n\ldots\omega_0}(X)|}{|h(f_{\omega_n\ldots\omega_0}(X))|}} = \lim_{n\to\infty} \frac{h'(b_n)}{h'(a_n)}.$$

Since h' is uniformly continuous with no zeros and since $|b_n - a_n| \to 0$ the last limit is equal to 1 which finishes the proof of the first part of our theorem.

In order to show the second part of this theorem it suffices to show that condition (a) of Theorem 2.2 is satisfied. So, let $\tau = \tau_0 \ldots \tau_{q-1}$ be an arbitrary word. Our aim is to show that $|(g_\tau)'(h(x_\tau))| = |(f_\tau)'(x_\tau)|$, where x_τ is the only fixed point of the map $f_\tau : X \mapsto X$. First notice that for every n

$$\frac{|g_{\tau^{n+1}\tau_0}(Y)|}{|g_{\tau^n\tau_0}(Y)|} = \frac{|g_{\tau^{n+1}\tau_0}(Y)|}{|g_{\tau^{n+1}}(Y)|} \cdot \frac{|g_{\tau^{n+1}}(Y)|}{|g_{\tau^n\tau_0\ldots\tau_{q-2}}(Y)|} \cdot \frac{|g_{\tau^n\tau_0\ldots\tau_{q-2}}(Y)|}{|g_{\tau^n\tau_0\ldots\tau_{q-3}}(Y)|} \cdot \ldots \cdot \frac{|g_{\tau^n\tau_0\tau_1}(Y)|}{|g_{\tau^n\tau_0}(Y)|}.$$

Hence

$$\lim_{n\to\infty} \frac{|g_{\tau^{n+1}\tau_0}(Y)|}{|g_{\tau^n\tau_0}(Y)|} = S^w_{\tau^\infty}(\tau_0) S^w_{\tau^\infty\tau_0\ldots\tau_{q-2}}(\tau_{q-1}) S^w_{\tau^\infty\tau_0\ldots\tau_{q-3}}(\tau_{q-2}) \ldots S^w_{\tau^\infty\tau_0}(\tau_1) \tag{2.5}$$

and similarly

$$\lim_{n\to\infty} \frac{|f_{\tau^{n+1}\tau_0}(X)|}{|f_{\tau^n\tau_0}(X)|} = S^w_{\tau^\infty}(\tau_0) S^w_{\tau^\infty\tau_0\ldots\tau_{q-2}}(\tau_{q-1}) S^w_{\tau^\infty\tau_0\ldots\tau_{q-3}}(\tau_{q-2}) \ldots S^w_{\tau^\infty\tau_0}(\tau_1). \tag{2.6}$$

Since $g_{\tau^{n+1}\tau_0}(Y) = g_\tau(g_{\tau_n\tau_0}(Y)$ and since $f_{\tau^{n+1}\tau_0}(X) = f_\tau(f_{\tau_n\tau_0}(X))$, it follows from the Mean Value Theorem that there exists $x_n \in f_{\tau_n\tau_0}(X)$ and $y_n \in g_{\tau_n\tau_0}(Y)$ such that $|g_{\tau^{n+1}\tau_0}(Y)| = |g'_\tau(y_n)|\cdot|g_{\tau_n\tau_0}(Y)|$ and $|f_{\tau^{n+1}\tau_0}(X)| = |f'_\tau(y_n)|\cdot|f_{\tau_n\tau_0}(X)|$. Thus in view of our assumptions and (2.5) and (2.6) we get

$$\lim_{n\to\infty}\frac{|g'_\tau(y_n)|}{|f'_\tau(x_n)|} = \lim_{n\to\infty}\frac{\frac{|g_{\tau^{n+1}\tau_0}(Y)|}{|g_{\tau^n\tau_0}(Y)|}}{\frac{|f_{\tau^{n+1}\tau_0}(X)|}{|f_{\tau^n\tau_0}(X)|}} = 1.$$

Now, a straightforward observation shows that $y_n \to y_\tau$ and $x_n \to x_\tau$, where y_τ and x_τ are fixed points of g_τ and f_τ respectively. Hence $|g'_\tau(y_\tau)| = |f'_\tau(x_\tau)|$ and equivalence of this condition with condition (1) of Theorem 2.1 finishes the proof. ■

One would like to improve the last theorem to the "if and only if" form, so we should point out here that in a special case when $\{f : X \to X : i \in \mathbb{N}\}$ and $\{g : Y \to Y : i \in \mathbb{N}\}$ are two non-linear conformal expanding repellers in $\mathbb{C}$, Przytycki and Urbański proved in [PU] that if conjugacy map h is bi-Lipschitz continuous, then h extends from X (or J_F) to a conformal homeomorphism on a neighbourhood of X. Additionally with the above assumption if the function $\{g : Y \to Y : i \in \mathbb{N}\}$ is real-analytic then the function $\{f : X \to X : i \in \mathbb{N}\}$ is real-analytic too.

CHAPTER 3

CONTINUED FRACTIONS

3.1 Computation of Hausdorff Measure

In this chapter we focus our attention on a special example of an infinite conformal iterated function system. We should point out that similar systems were introduced and studied in [GM] and [MU1]. Now, let X be a closed disc on a complex plane centered at the point $1/2$ with radius $1/2$, and let $V = B(1/2, 3/4)$. Given $k \geq 1$, set $I = I_k = \{n + ki \ : \ n \in \mathbb{N}\}$, and for every index $n + ki \in I$ define $\phi_{n+ki} : V \to V$ by

$$\phi_{n+ki}(t) = \frac{1}{n + ki + t}.$$

One can easily verify that for every positive integer n, $\phi_{n+ki}(X) \subseteq X$ and $\phi_{n+ki}(V) \subseteq V$, or even more precisely

$$\phi_{n+ki}(B(1/2, 1/2)) = \frac{1}{B(n + 1/2 + ki, 1/2)}. \tag{3.1}$$

Moreover we have that $\phi'_{n+ki}(t) = -(n + ki + t)^{-2}$, and hence $||\phi'_{n+ki}|| = |n + ki|^{-2} < (1 + k^2)^{-1} < 1$. That gives us the universal contractive constant from the definition of iterated function system. It is also easy to check that our system is conformal, first three conditions from the definition are trivially satisfied, and the (BDP) is a straightforward consequence of the Koebe distortion theorem. More specifically one can deduce from the Koebe theorem that there exists a constant $K \geq 1$ such that for every $n+ki \in I_k$

$$\frac{1}{K|n+ki|^2} \le |\phi'_{n+ki}(z)| \le \frac{K}{|n+ki|^2}.$$

As announced in the introduction we want to turn our attention to the limit set $J = J_k$ associated with the system. In particular we want to investigate the Hausdorff dimension, and then the h-dimensional Hausdorff and packing measures of this set, where $h = \dim_H(J_k)$.

We start our investigation with the following lemma.

Lemma 3.1 $\lim_{k\to\infty} \dim_H(J_k) = 1/2$.

Proof According to Theorem 1.1, $\dim_H(J_k) = \inf\{t \ge 0 : \; P(t) \le 0\}$. One can prove using the chain rule that the sequence $\psi_n(t)$ is subadditive, that is

$$K^{-t}\psi_n(t)\psi_m(t) \le \psi_{n+m}(t) \le \psi_n(t)\psi_m(t) \le \psi_1(t)^{n+m}. \tag{3.2}$$

Therefore the fact that $P(t) = \infty$ is equivalent to saying that $\psi_1(t) = \infty$. In our case,

$$\psi_1(t) = \sum_{n=1}^{\infty} ||\phi'_{n+ki}||^t = \sum_{n=1}^{\infty} \frac{1}{|n+ki|^{2t}}.$$

In particular $\psi_1(1/2) = \sum_{n=1}^{\infty} |n+ki|^{-1} = \infty$. This proves that the system is regular and $\dim_H(J_k) > 1/2$ for all k.

Fix $\epsilon > 0$, and choose k, depending on ϵ, so large that

$$\psi_1\Big(\frac{1}{2}+\epsilon\Big) = \sum_{n=1}^{\infty} \frac{1}{|n+ki|^{1+\epsilon/2}} < 1.$$

Then using the subadditive property (3.2) we obtain

$$P\Big(\frac{1}{2}+\epsilon\Big) \;\; = \;\; \lim_{m\to\infty} \frac{1}{m} \log \psi_m\Big(\frac{1}{2}+\epsilon\Big) \le \lim_{m\to\infty} \frac{1}{m} \log \psi_1\Big(\frac{1}{2}+\epsilon\Big)^m$$

$$= \log \psi_1\Big(\frac{1}{2}+\epsilon\Big) < 0.$$

We get that $\dim_H(J_k) \le 1/2+\epsilon$. When we let $\epsilon \searrow 0$, which implies $k \to \infty$, our proof is finished. ■

Remark 3.2 *Notice that if $k = 0$ we have a system of real continued fractions $\phi_n(x) = (n+x)^{-1}$, for which the limit set J_0 is the unit interval without rational numbers. Obviously in this case the Hausdorff dimension of the limit set is equal to 1, and both 1-dimensional Hausdorff measure of J_0 and 1-dimensional packing measure of J_0 are 1.*

Theorem 3.3 *Let k be such that $1/2 < h = \dim_H(J_k) < 1$. Then $\mathcal{H}^h(J_k) = 0$.*

Proof Let m be the conformal measure associated to our conformal iterated function system. The idea of the proof is based on the following fact (Lemma 4.9 in [MU1]): If S is a regular conformal iterated function system and there exists a sequence of points $z_j \in X(\infty)$ and a sequence of positive reals $r_j,\ j \ge 1$, such that $r_j \to 0$ and

$$\limsup_{j\to\infty} \frac{m(B(z_j, r_j))}{r_j^h} = \infty,$$

then $\mathcal{H}^h(J_k) = 0$.

In our case, $z_j = 0$ for every j, since 0 is the only point in $X(\infty)$. Hence it is sufficient to show that

$$\overline{\lim_{r\to 0}} \frac{m(B(0,r))}{r^h} = \infty. \tag{3.3}$$

Notice that

$$\begin{aligned}
m(B(0,r)) &\geq \sum_{\phi_{n+ki}(X)\subseteq B(0,r)} m(\phi_{n+ki}(X)) \\
&= \sum_{\phi_{n+ki}(X)\subseteq B(0,r)} \int_X |\phi_{n+ki}|^h \, dm \\
&\geq \sum_{\phi_{n+ki}(X)\subseteq B(0,r)} K^{-h}||\phi'_{n+ki}||^h \\
&= K^{-h} \sum_{\phi_{n+ki}(X)\subseteq B(0,r)} |n+ki|^{-2h} \\
&\asymp K^{-h} \sum_{\phi_{n+ki}(X)\subseteq B(0,r)} n^{-2h},
\end{aligned}$$

where by $A \asymp B$ we mean that there exists some constant $C \geq 1$ such that $C^{-1} \leq A/B \leq C$.

We have to find out for how many n, $B(0,r) \cap \phi_{n+ki}(X) \neq \emptyset$ or for which n, $\phi_{n+ki}(X) \subset B(0,r)$. Now,

$$\begin{aligned}
y \in B(0,r) \cap \phi_{n+ki}(X) &\Leftrightarrow |y| < r \text{ and } y \in \frac{1}{B(n+1/2+ki, 1/2)} \\
&\Leftrightarrow \frac{1}{|y|} > \frac{1}{r} \text{ and } \frac{1}{y} \in B(n+1/2+ki, 1/2).
\end{aligned}$$

Hence $B(0,r) \cap \phi_{n+ki}(X) \neq \emptyset$ if and only if $B(n+1/2+ki, 1/2)$ contains a complex number of modulus bigger than $1/r$.

Notice that if z is a point in $\overline{B(n+1/2+ki, 1/2)}$ of maximal possible modulus, then

$$|z| = \sqrt{n^2+n+\frac{1}{4}+k^2} + \frac{1}{2}.$$

Therefore $B(0,r) \cap \phi_{n+ki}(X) \neq \emptyset$ only if

$$\sqrt{n^2+n+k^2+\frac{1}{4}} + \frac{1}{2} > \frac{1}{r}.$$

Let $A = \{n | \sqrt{n^2+n+k^2+1/4}+1/2 > 1/r\}$, and let $n(r)$ be the minimal element of A. One can see that $n(r) \asymp 1/r$ by the minimality of $n(r)$. Using the integral test we are able to evaluate the limit introduced at the beginning of the proof:

$$\begin{aligned}
\overline{\lim_{r\to 0}}\frac{m(B(0,r))}{r^h} &\geq \overline{\lim_{r\to 0}}\frac{K^{-h}\sum_{n\geq n(r)+1} n^{-2h}}{r^h} \asymp \overline{\lim_{r\to 0}}\frac{K^{-h}\int_{n(r)}^{\infty} x^{-2h}\,dx}{r^h} \\
&= \overline{\lim_{r\to 0}}\frac{n(r)^{1-2h}}{(2h-1)K^h r^h} \asymp \overline{\lim_{r\to 0}}\frac{r^{2h-1}}{(2h-1)K^h r^h} \\
&= \overline{\lim_{r\to 0}}\frac{r^{h-1}}{(2h-1)K^h} = \infty.
\end{aligned}$$

We conclude that $\mathcal{H}^h(J_k) = 0$, which completes the proof. ∎

3.2 Estimation of Packing Measure

Now we turn our attention to the h-dimensional packing measure of the limit set J_k. We begin with a simple lemma.

Lemma 3.4 *If f is the function defined on the complex plane by $f(z) = 1/z$ and $C(x,r)$ is the circle centered at x and of radius r, then*

$$f(C(x,r)) = C\Big(\frac{\overline{x}}{|x|^2-r^2}, \frac{r}{||x|^2-r^2|}\Big). \tag{3.4}$$

Proof One should observe that for $\lambda > 0,\ \lambda \neq 1$ the equation

$$\left|\frac{z-p}{z-q}\right| = \lambda$$

represents the circle with respect to which the points p and q are symmetric, see [P]. Moreover the center and the radius of this circle are given by the formulas

$$x = \frac{p-\lambda^2 q}{1-\lambda^2}, \quad r = \lambda\frac{|p-q|}{|1-\lambda^2|}.$$

Then a straightforward computation proves the claim of the lemma. ■

Theorem 3.5 *Let k be such that $1/2 < h < 1$. Then $0 < \mathcal{P}^h(J_k) < \infty$.*

Proof It is a general fact that if the limit set J of a conformal iterated function system has nonempty intersection with an interior of the set X, then the packing measure of this set is always positive. Hence we only need to show that the packing measure of J_k is finite.

Following Theorem 2.5 from [MU2] we must prove that there exist three constants $L > 0$, $\xi > 0$, and $\gamma \geq 1$, and a finite set F, such that for all $n \in I \setminus F$ and for all r with $\gamma \operatorname{diam}(\phi_{n+ki}(X)) < r \leq \xi$ there is some $x \in \phi_{n+ki}(X)$ such that

$$\frac{m(B(x,r))}{r^h} \geq L. \tag{3.5}$$

According to the result proven in the lemma above the diameter of the ball $\phi_{p+ki}(X)$ is

$$\operatorname{diam}(\phi_{p+ki}(X)) = 2\frac{1/2}{||p+1/2+ki|^2 - 1/4|} \asymp \frac{1}{p^2}, \tag{3.6}$$

for p large enough. Let Γ_p denote the arc that is the image of the half-line $pt+ki,\ t \geq 1$, under the function f. One can see that two the most distant points of Γ_p lying in $\phi_{p+ki}(X)$ are $(p+1+ki)^{-1}$ and $(p+ki)^{-1}$. Therefore we can choose x to be the point in $\Gamma_p \cap \phi_{p+ki}(X)$ with $|x| = 1/(p+1)$, if only p is large enough. Additionally, set $\xi = 1$ and $\gamma = 8$. To prove the theorem we only have to find $L > 0$ such (3.5) holds. We consider two separate cases.

Case 1. Suppose that $|x| \leq r$. In this case the ball $B(x,r)$ contains infinitely many balls of the form $\phi_{n+ki}(X)$; in fact it contains all the balls for n greater than some n_0. On the arc Γ_p choose a point y such that $\rho(x,y) = r$. Let l denote the length of a part of Γ_p from 0 to y. There exists a unique $1 \leq m \leq p$ so that

$$\frac{m}{p+1} \leq r \leq \frac{m+1}{p+1}.$$

Simple geometry gives us that

$$\begin{aligned} |y| &> \frac{1}{\pi} l > \frac{1}{\pi}\Big(\frac{1}{p+1} + r\Big) > \frac{1}{\pi}\frac{m+1}{p+1} \\ &= \frac{1}{\frac{\pi(p+1)}{m+1}} \geq \frac{1}{8\lfloor p/m \rfloor} \end{aligned}$$

The above computation tells us that

$$\frac{1}{B(8\lfloor p/m \rfloor + 1/2 + ki, 1/2)} \cap B(x,r) \neq \emptyset, \tag{3.7}$$

so in other words we can assume that $n_0 = 8\lfloor p/m \rfloor$. We have that

$$\begin{aligned} m(B(x,r)) &\geq \sum_{n=8\lfloor \frac{p}{m} \rfloor + 1}^{\infty} n^{-2h} \asymp \int_{8\lfloor \frac{p}{m} \rfloor}^{\infty} x^{-2h}\, dx \\ &= \frac{1}{2h-1} \frac{1}{(8\lfloor \frac{p}{m} \rfloor)^{2h-1}} = \frac{1}{8^{2h-1}(2h-1)} \frac{1}{\lfloor \frac{p}{m} \rfloor^{2h-1}} \\ &\geq \frac{1}{8^{2h-1}(2h-1)} \frac{1}{\lfloor \frac{p}{r(p+1)} \rfloor^{2h-1}} > \frac{1}{8^{2h-1}(2h-1)} \frac{1}{(\frac{1}{r})^{2h-1}} \\ &= \frac{1}{8^{2h-1}(2h-1)} r^{2h-1} \geq \frac{1}{8^{2h-1}(2h-1)} r^{h}. \end{aligned}$$

Case 2. Assume that $r < |x|$. In this case the ball $B(x,r)$ contains only finitely many balls of the form $\phi_{n+ki}(X)$. It certainly contains the pth ball, so all we have to do is to find the maximum index l so that $\phi_{l+ki}(X) \subset B(x,r)$. Then there exists a unique $l \geq p$ such that

$$\sum_{n=p}^{l} \operatorname{diam}\Big(\frac{1}{B(n+\frac{1}{2}+ki,\frac{1}{2})}\Big) \leq r \leq \sum_{n=p}^{l+1} \operatorname{diam}\Big(\frac{1}{B(n+\frac{1}{2}+ki,\frac{1}{2})}\Big)$$

The formula (3.6) for the diameter of $\phi_{n+ki}(X)$ immediately gives that for n large enough

$$\frac{1}{(n+1)^2} < \operatorname{diam}\Big(\frac{1}{B(n+1/2+ki,1/2)}\Big) < \frac{1}{n^2} \tag{3.8}$$

Hence, using the integral test, we obtain

$$\begin{aligned}
\frac{1}{p+1} - \frac{1}{l+2} &= \int_{p+1}^{l+2} x^{-2}\, dx \leq \sum_{n=p+1}^{l+1} \frac{1}{n^2} < r \\
&< \sum_{n=p}^{l+1} \frac{1}{n^2} \leq \int_{p-1}^{l+1} x^{-2}\, dx = \frac{1}{p-1} - \frac{1}{l+1}.
\end{aligned}$$

This shows that

$$\frac{1}{p+1} - \frac{1}{l+2} \leq r \leq \frac{1}{p-1} - \frac{1}{l+1} \tag{3.9}$$

Applying the Mean Value Theorem we get

$$\begin{aligned}
m(B(x,r)) &\geq \sum_{n=p}^{l} n^{-2h} \asymp \int_{p}^{l} x^{-2h}\, dx \\
&= \frac{1}{2h-1}\Big(\Big(\frac{1}{p}\Big)^{2h-1} - \Big(\frac{1}{l}\Big)^{2h-1}\Big) \\
&\geq \frac{1}{2h-1}\Big(\frac{1}{p} - \frac{1}{l}\Big)(2h-1)\Big(\frac{1}{z}\Big)^{2h-2} \\
&= \Big(\frac{1}{p} - \frac{1}{l}\Big)\Big(\frac{1}{z^2}\Big)^{h-1},
\end{aligned}$$

for some $p \leq z \leq l$. Recall that $r \geq 8/p^2$, which implies $z^{-2} \leq r/8$. Hence, using (3.9)

$$m(B(x,r)) \;\; \geq \;\; \Big(\frac{1}{p} - \frac{1}{l}\Big)\Big(\frac{r}{8}\Big)^{h-1}$$

$$
\begin{aligned}
&= \Big[\Big(\frac{1}{p} - \frac{1}{p-1}\Big) + \Big(\frac{1}{p-1} - \frac{1}{l+1}\Big) + \Big(\frac{1}{l+1} - \frac{1}{l}\Big)\Big] 8^{1-h} r^{h-1} \\
&\geq \Big[- \frac{1}{p(p-1)} + r - \frac{1}{l(l+1)}\Big] 8^{1-h} r^{h-1} \\
&\geq \Big[r - \frac{2}{p(p-1)}\Big] 8^{1-h} r^{h-1} \\
&\geq \Big[r - \frac{4}{p^2}\Big] 8^{1-h} r^{h-1} \geq \Big[r - \frac{r}{2}\Big] 8^{1-h} r^{h-1} = \frac{1}{2} 8^{1-h} r^{h}.
\end{aligned}
$$

Choosing L to be $\min\{8^{1-2h}(2h-1)^{-1},\ 2^{-1}8^{1-h}\}$ completes the proof. ■

3.3 Additional Remarks

According to the first equality of Theorem 1.1, and since the Hausdorff dimension of the limit set of the system $\{\phi_{n+ki} : n \geq 1, k \in \mathbb{Z}\}$ is greater than 1, we can add to the family $\Phi_l = \{\phi_{n+li} : n \geq 1\}$, $l \geq 1$, finitely many mappings from $\{\phi_{n+ki} : n \geq 1, k \in \mathbb{Z}\}, k \neq l$ to obtain the systems whose limit sets J have Hausdorff dimensions greater than 1. Then, employing methods similar to those used in the proofs of Theorems 3.3 and 3.5, we find that $0 < \mathcal{H}^h(J) < \infty$ and $\mathcal{P}^h(J) = \infty$.

Here instead of using Lemma 4.9 from [MU1] and one should apply Lemma 4.11 from [MU1] which states

Lemma 3.6 *Let $S = \{\phi_i : i \in I\}$ be a regular conformal iterated function system. Suppose that there are two constants $L > 0, \gamma \geq 1$ such that for every $i \in I$ and every $r \geq \gamma diam(\phi_i(X))$ there exists $y \in \phi_i(V)$ such that $m(B(y,r)) \leq Lr^h$. Then $\mathcal{H}^h(J) > 0$.*

and Theorem 2.5 from [MU2] one should be replaced with Theorem 2.6 from [MU2], which says

Theorem 3.7 *Let $S = \{\phi_i : i \in I\}$ be a regular conformal iterated function system. Suppose that there exists a subset $\emptyset \neq Z \subset X(\infty)$ such that for every $z \in Z$ there exists $i(z) \in I$ and a set $R(z) \subset (0, dist(X, \partial V))$ such that*

(a) $\phi_{i(z)}(B(z, \sup R(z)) \cap J) = \phi_{i(z)}(B(z, \sup R(z))) \cap J$,

(b) $\phi_{i(z)}(B(z, \sup R(z))) \subset X$,

(c) $\inf\left\{\frac{m(B(z,r))}{r^h} : z \in Z, r \in R(z)\right\} = 0.$

Then $\mathcal{P}^h(J) = \infty$.

Let us remark that this fact distinguishes these systems from the full family $\{\phi_{n+ki} : n \geq 1, k \in \mathbb{Z}\}$ and the families investigated in this chapter, since for them the packing measure is finite and positive, whereas here the Hausdorff measure is zero.

We finish this section with two examples that relate to the results established in Chapter 2.

Example No two different systems of bounded geometry of real continued fractions are bi-Lipschitz conjugate. We prove it by the method of contradiction. Let $X = [0, 1]$ and let $S = \{\phi_n : X \to X,\ n \in \mathbb{N}\}$ be a system of real continued fractions, that is

$$\phi_n(x) = \frac{1}{n + x}.$$

Suppose that $F = \{f_i : X \to X, i \in \mathbb{N}\}$ and $G = \{g_i : X \to X, i \in \mathbb{N}\}$ are two subsystems of S of bounded geometry that are bi-Lipschitz conjugate and that $F \neq G$. If $h : J_F \to J_G$ is a conjugating function then there exists an index i such that $h \circ f_i = g_i \circ h$ and $f_i \neq g_i$. We can assume that $f_i = \phi_n$ and $g_i = \phi_k$ with $n > k$. Theorem 2.2 and Theorem 2.1 imply $|\phi_k'(x_k)| = |\phi_n'(x_n)|$, where x_k and x_n are the only fixed points of $\phi_k : X \to X$ and $\phi_n : X \to X$ respectively. However, a straightforward computation shows that

$$x_n = \frac{\sqrt{n^2+4}-n}{2} \quad \text{and} \quad x_k = \frac{\sqrt{k^2+4}-k}{2}.$$

Hence, $|\phi_n'(x_n)| = 2(n^2+2+n\sqrt{n^2+4})^{-1} < 2(n^2+2+n\sqrt{n^2+4})^{-1} = |\phi_k'(x_k)|$, which is a contradiction.

Example The above property is not true for the systems of complex continued fractions. Namely, let X be a ball defined at the beginning of this chapter. Fix k, and consider two systems: $F = \{\phi_{n+ki} : X \to X,\ n \in \mathbb{N}\}$ and $G = \{\phi_{n-ki} : X \to X,\ n \in \mathbb{N}\}$. Define $h : \mathbb{C} \to \mathbb{C}$ by $h(z) = \overline{z}$. Then h establishes a bi-Lipschitz (analytical) conjugacy between systems F and G, since $h \circ \phi_{n+ki}(z) = \phi_{n-ki} \circ h(z)$ for all n.

3.4 Numerical Results

It was shown in [MU1] that the limit set J related to the system where there is no restriction for an index k ($k \in \mathbb{Z}$ arbitrary) has the following properties:

$$1.2484 < h = \dim_H(J) < 1.9,$$

$$\mathcal{H}^h(J) = 0,$$

$$0 < \mathcal{P}^h(J) < \infty.$$

We want to improve the lower estimation for h. For that reason we directly state the following theorem from [MU1].

Theorem 3.8 *For any real number $t \geq 0$ the following three conditions are equivalent:*

(a) $t = h = \dim_H(J)$;

(b) t is the only number such that $1 \leq \psi_n(t) \leq K^d$ for all $n \geq 1$, where d is the dimension of the Euclidean space containing X;

(c) t is the only number such that $\tilde{\psi}_n(t) \leq 1 \leq \psi_n(t)$ for all $n \geq 1$, where $\tilde{\psi}_n(t) = \sum_{\omega \in I^n} \inf |\phi'_\omega|^t$ and $\inf |\phi'_\omega| = \inf\{|\phi'_\omega(x)| : \ x \in X\}$.

Let $S = \{\phi_i : \ i \in I\}$ be the full system of complex continued fractions. We set $\tilde{t}_n$ to be the only solution of the equation $\tilde{\psi}_n(t) = 1$. Hence,

$$\sum_{\omega \in I^n} \inf |\phi'_\omega|^{\tilde{t}_n} = 1,$$

and it is obvious that $\tilde{t}_n \leq h$ for all $n \geq 1$.

If F is a finite subset of I, define $\tilde{t}_n^F$ to be the only number such that

$$\sum_{\omega \in F^n} \inf |\phi'_\omega|^{\tilde{t}_n^F} = 1. \tag{3.10}$$

It is clear that for every $F \subset I$ finite $\tilde{t}_n^F \leq \tilde{t}_n \leq h$ for all $n \geq 1$, and moreover if $F_1 \subset F_2$ then $\tilde{t}_n^{F_1} \leq \tilde{t}_n^{F_2}$. Every map in the system S is a Möbius map, namely

$$\phi_i(z) = \frac{a_i z + b_i}{c_i z + d_i},$$

with $a_i = 0$, $b_i = c_i = 1$, and $d_i = n + ki$, where $n \in \mathbb{N}$, $k \in \mathbb{Z}$. Additionally $a_i d_i - b_i c_i = -1$, so if we compose several maps of this form we will obtain that for each $\omega \in I^n$

$$\phi_\omega(z) = \frac{a_\omega z + b_\omega}{c_\omega z + d_\omega},$$

where $a_\omega,\ b_\omega,\ c_\omega,$ and d_ω are complex numbers of the form $n+ki,\ n \in \mathbb{N},\ k \in \mathbb{Z}$, and also $a_\omega d_\omega - b_\omega c_\omega = (-1)^n$. Hence,

$$\phi'_\omega(z) = \frac{a_\omega(c_\omega z + d_\omega) - (a_\omega z + b_\omega)c_\omega}{(c_\omega z + d_\omega)^2} = \frac{(-1)^n}{(c_\omega z + d_\omega)^2}$$

and

$$|\phi'_\omega(z)| = \frac{1}{|c_\omega z + d_\omega|^2} = \frac{1}{|c_\omega|^2 \left| z + \frac{d_\omega}{c_\omega} \right|^2}.$$

Since $z \in B(\frac{1}{2}, \frac{1}{2})$, we have $z + \frac{d_\omega}{c_\omega} \in B(\frac{1}{2} + \frac{d_\omega}{c_\omega}, \frac{1}{2})$. The point of maximal modulus in this ball is the further (from the origin) of two points lying on the circle and the line through 0 and $\frac{1}{2} + \frac{d_\omega}{c_\omega}$; that is the point

$$z_{\max} = \left(1 + \frac{1}{|1 + \frac{2d_\omega}{c_\omega}|}\right)\left(\frac{1}{2} + \frac{d_\omega}{c_\omega}\right),$$

and also

$$\left| z_{\max} + \frac{d_\omega}{c_\omega} \right|^2 = \frac{1}{4}\left(\left| 1 + \frac{2d_\omega}{c_\omega} \right| + 1 \right)^2.$$

Therefore,

$$\inf |\phi'_\omega| = \frac{4}{(|c_\omega + 2d_\omega| + |c_\omega|)^2}.$$

One can compute directly the values of c_ω and d_ω for a given ω. Being restricted by computer power we let $|\omega| = 4$ and $F = \{n+ki :\ 1 \le n \le 13,\ -6 \le k \le 6\}$. Then running a computer program we approximate the solution of the equation (3.10) with an accuracy of 2^{-15} and find that

$$1.71518 \leq \tilde{t}_4^F \leq 1.71536.$$

Hence $1.71518 \leq h = \dim_H(J)$, which substantially improves the previous result.

CHAPTER 4

POSITIVE RECURRENT FUNCTIONS

4.1 A New Class of Functions

In this chapter, using the concept of an iterated function system, we construct a new, natural class of positively recurrent functions and we show that they have stronger properties than the general functions considered by Sarig in [Sa]. Our exposition is similar to the approach developed in [HMU, MU1, U, and Wa].

To begin with, let $I = \mathbb{N}$ (an index set for the iterated function system) be the set of positive integers and let $\Sigma = \mathbb{N}^{\infty}$ be the space of infinite words equipped with the product topology. Let $\sigma : \Sigma \to \Sigma$ be the shift transformation (cutting out the first coordinate), $\sigma(\{x_n\}_{n=1}^{\infty}) = (\{x_n\}_{n=2}^{\infty})$. Fix $\beta > 0$. If $\phi : \Sigma \to \mathbb{R}$ and $n \geq 1$, we set

$$V_n(\phi) = \sup\{|\phi(x) - \phi(y)| : x_1 = y_1, x_2 = y_2, \ldots, x_n = y_n\}.$$

The function ϕ is said to be Hölder continuous of order β if and only if

$$V(\phi) = \sup_{n\geq 1}\{e^{\beta n} V_n(\phi)\} < \infty. \tag{4.1}$$

We also assume that

$$\sup_{\omega\in\Sigma} \sum_{\tau\in\sigma^{-1}(\omega)} e^{\phi(\tau)} < \infty. \tag{4.2}$$

This assumption allows us to introduce the Perron-Frobenius-Ruelle operator L_ϕ :

$C_b(\Sigma) \to C_b(\Sigma)$,

$$L_\phi(g)(\omega) = \sum_{\tau \in \sigma^{-1}(\omega)} e^{\phi(\tau)} g(\tau)$$

acting on $C_b(\Sigma)$, the space of all bounded continuous real-valued functions on Σ equipped with the norm $||.||_0$, where $||k||_0 = \sup_{x \in \Sigma} |k(x)|$. Moreover,

$$||L_\phi||_0 \le L_\phi(1) = \sup_{\omega \in \Sigma} \sum_{\tau \in \sigma^{-1}(\omega)} e^{\phi(\tau)} < \infty.$$

We extend the standard definition of topological pressure by setting

$$P(\phi) = \lim_{n \to \infty} \frac{1}{n} \log \Big(\sum_{|\omega|=n} \sup_{\tau \in [\omega]} \exp \Big(\sum_{j=0}^{n-1} \phi \circ \sigma^j(\tau) \Big) \Big), \tag{4.3}$$

where $[\omega] = \{\rho \in \Sigma : \rho_1 = \omega_1, \rho_2 = \omega_2, \dots, \rho_{|\omega|} = \omega_{|\omega|}\}$. Notice that the limit exists since the partition functions

$$Z_n(\phi) = \sum_{|\omega|=n} \sup_{\tau \in [\omega]} \exp \Big(\sum_{j=0}^{n-1} \phi \circ \sigma^j(\tau) \Big)$$

form a submultiplicative sequence. Notice also that our definition of pressure formally differs from that provided by Sarig in [Sa] which reads that given $i \in \mathbb{N}$

$$P(\phi) = \lim_{n \to \infty} \frac{1}{n} \log Z_n(\phi, i), \tag{4.4}$$

where

$$Z_n(\phi, i) = \sum \exp \Big(\sum_{j=0}^{n-1} \phi \circ \sigma^j(\omega) \Big)$$

and the summation is taken over all elements ω satisfying $\sigma^n(\omega) = \omega$ and $\omega_1 = i$. However in [Sa] Sarig proves Theorem 2 which says that $P(\phi) = \sup\{P(\phi|_Y)\}$, where the supremum is taken over all topologically mixing subshifts of finite type $Y \subset \Sigma$

and the same proof goes through with (4.4) replaced by (4.3) (comp. Theorem 3.1 of [MU2]). Thus we have the following.

Lemma 4.1 *The definitions of topological pressures given by (4.3) and (4.4) coincide.*

Proof The following direct proof of this lemma was presented to us by Sarig. Fix $i \in \mathbb{N}$. Using Hölder continuity of the function ϕ we can write

$$Z_n(\phi) \asymp \sum_{|\omega|=n} \exp\Big(\sum_{j=0}^{n-1} \phi(\omega^\infty)\Big) \asymp \sum_{|\omega|=n} \exp\Big(\sum_{j=0}^{n} \phi((i\omega)^\infty)\Big) = Z_{n+1}(\phi, i).$$

Thus the lemma is proven. ■

Following [Sa] we call the function $\phi : \Sigma \to \mathbb{R}$ *positive recurrent* if for every $i \in \mathbb{N}$ there exists a constant M_i and an integer N_i such that for all $n \geq N_i$

$$M_i^{-1} \leq Z_n(\phi, i)\lambda^{-n} \leq M_i$$

for some $\lambda > 0$.

As we already have said the main purpose of this section is to provide a wide natural class of examples of positive recurrent potential which additionally satisfy much stronger properties than those claimed in Theorem 4 of [Sa].

Let now $\phi^{(i)} : X \to \mathbb{R}$, $i \in \mathbb{N}$, be a family of continuous functions such that

$$\sup_X \sum_{i \in \mathbb{N}} e^{\phi^{(i)}(x)} < \infty. \tag{4.5}$$

We define a function $\phi : \Sigma \to \mathbb{R}$ by setting

$$\phi(\omega) = \phi^{(\omega_1)}(\pi(\sigma(\omega))). \tag{4.6}$$

It easily follows from (4.5) that $P(\phi) < \infty$.

Define first an auxiliary Perron-Frobenius operator $\tilde{L}_\phi : C(X) \to C(X)$ given by the formula

$$\tilde{L}_\phi(g)(x) = \sum_{i \in \mathbb{N}} e^{\phi^{(i)}(x)} g(\phi_i(x)).$$

$\tilde{L}_\phi$ is continuous, positive and $||\tilde{L}_\phi||_0 \le \sup_X \sum_{i\in\mathbb{N}} e^{\phi^{(i)}(x)} < \infty$. Let $\tilde{L}_\phi^* : C(X)^* \to C(X)^*$ be the conjugate operator and following Bowen's approach from [Bo] consider the map

$$\mu \mapsto \frac{\tilde{L}_\phi^*(\mu)}{\tilde{L}_\phi^*(\mu)(1)}.$$

of the space of Borel probability measures on X into itself. This map is continuous in the weak-$*$ topology of measures and therefore, in view of the Schauder-Tichonov theorem, it has a fixed point, say m_ϕ. Thus

$$\tilde{L}_\phi^*(m_\phi) = \lambda m_\phi \tag{4.7}$$

with $\lambda = \tilde{L}_\phi^*(m_\phi)(1)$.

Now we should state our main theorem.

Theorem 4.2 *Suppose that the function $\phi : \Sigma \to \mathbb{R}$ defined by (4.6) and satisfying (4.5) is Hölder continuous. Let L_ϕ^* be the operator conjugate to L_ϕ. Then ϕ is positive recurrent with $\lambda = e^{P(\phi)}$. Moreover there exists $M > 0$ such that $M^{-1} \le \lambda^{-n} L_\phi^n(1) \le M$ for all $n \ge 1$. Suppose additionally that $\phi_i(X) \cap \phi_j(X) = \emptyset$ for all $i, j \in \mathbb{N}$, $i \neq j$. Then there are a probability measure ν on Σ and a bounded away from zero and infinity, Hölder continuous function $h : \Sigma \to (0, \infty)$ such that $L_\phi^*(\nu) = \lambda\nu$,*

$L_\phi(h) = \lambda h$, $\nu(h) = 1$ and $\lambda^{-n} L_\phi^n(g) \to (\int g\, d\nu)h$ uniformly for every uniformly continuous bounded function g. Additionally $\lambda^{-n} L_\phi^n(g) \to (\int g\, d\nu)h$ exponentially fast for each Hölder continuous bounded function g.

4.2 Proof of Theorem 4.2

To prove Theorem 4.2 we will be using the methods of thermodynamic formalism. Given $n \geq 1$ and $\omega \in \mathbb{N}^n$, define

$$S_\omega(\phi) = \sum_{j=1}^{n} \phi^{(\omega_j)} \circ \phi_{\sigma^j \omega}.$$

Let us then prove the following.

Lemma 4.3 *If $x, y \in \phi_\tau(X)$ for some $\tau \in \mathbb{N}^*$, then for all $\omega \in \mathbb{N}^*$*

$$|S_\omega(\phi)(x) - S_\omega(\phi)(y)| \leq \frac{V(\phi)}{1 - e^{-\beta}} e^{-\beta|\tau|}$$

Proof Let $n = |\omega|$. Write $x = \phi_\tau(u)$, $y = \phi_\tau(w)$, where $u, w \in X$. By (4.1) we get

$$\begin{aligned}
\Big|\sum_{j=1}^{n} \phi^{(\omega_j)}(\phi_{\sigma^j\omega}(x)) &- \sum_{j=1}^{n} \phi^{(\omega_j)}(\phi_{\sigma^j\omega}(y))\Big| \\
&= \left|\sum_{j=1}^{n} \phi^{(\omega\tau)_j} \circ \phi_{\sigma^j\omega\tau}(u) - \sum_{j=1}^{n} \phi^{(\omega\tau)_j} \circ \phi_{\sigma^j\omega\tau}(w)\right| \\
&\leq \sum_{j=1}^{n} \left|\phi^{(\omega\tau)_j} \circ \phi_{\sigma^j\omega\tau}(u) - \phi^{(\omega\tau)_j} \circ \phi_{\sigma^j\omega\tau}(w)\right| \\
&\leq \sum_{j=1}^{n} V(\phi) e^{-\beta(n+|\tau|-j)} \\
&\leq \frac{V(\phi)}{1 - e^{-\beta}} e^{-\beta|\tau|}
\end{aligned}$$

The proof is finished. ■

Remark 4.4 *We allow in Lemma 4.3 τ to be the empty word $\emptyset$. Then $\phi_\emptyset = Id_X$ and $|\emptyset| = 0$.*

Set

$$Q = \exp\Big(V(\phi)\frac{e^{-\beta}}{1-e^{-\beta}}\Big).$$

We shall prove the following.

Lemma 4.5 *The eigenvalue λ (see 4.7) of the dual Perron-Frobenius operator is equal to $e^{P(\phi)}$.*

Proof Iterating (4.7) we get

$$\begin{aligned}\lambda^n &= \lambda^n m_\phi(1) = \tilde{L}_\phi^{*n}(1) = \int_X \tilde{L}_\phi^n(1)\, dm_\phi \\ &= \int_X \sum_{|\omega|=n} \exp(S_\omega(\phi)(x))\, dm_\phi \le \sum_{|\omega|=n} ||\exp(S_\omega(\phi))||_0,\end{aligned}$$

where $||.||_0$ denotes the supremum norm. So,

$$\log\lambda \le \lim_{n\to\infty}\frac{1}{n}\log\Big(\sum_{|\omega|=n} ||\exp(S_\omega(\phi))||_0\Big) = P(\phi).$$

Fix now $\omega \in \mathbb{N}^n$ and take a point x_ω where the function $S_\omega(\phi)$ takes on its maximum. In view of Lemma 4.3, for every $x \in X$ we have

$$\sum_{|\omega|=n} \exp(S_\omega(\phi)(x)) \ge Q^{-1}\sum_{|\omega|=n} \exp(S_\omega(\phi)(x_\omega)) = Q^{-1}\sum_{|\omega|=n} ||\exp(S_\omega(\phi))||_0.$$

Hence, iterating (4.7) as before,

$$\lambda^n = \int_X \sum_{|\omega|=n} \exp(S_\omega(\phi))\, dm_\phi \ge Q^{-1}\sum_{|\omega|=n} ||\exp(S_\omega(\phi))||_0.$$

So, $\log \lambda \geq \lim_{n\to\infty} \frac{1}{n} \log\Big(\sum_{|\omega|=n} ||\exp(S_\omega(\phi))||_0\Big) = P(\phi)$, which is enough to complete the proof. ■

Let $\tilde{L}_0$ and L_0 denote the corresponding normalized Perron-Frobenius operators, i.e. $\tilde{L}_0 = e^{-P(\phi)}\tilde{L}_\phi$ and $L_0 = e^{-P(\phi)}L_\phi$. We shall prove the following.

Theorem 4.6 $m_\phi(J) = 1$.

Proof Since by (4.7)

$$\tilde{L}_0^*(m_\phi) = m_\phi \tag{4.8}$$

and consequently $\tilde{L}_0^{*n}(m_\phi) = m_\phi$ for all $n \geq 0$, we have

$$\int_X L_0^n f\, dm_\phi = \int_X \sum_{|\omega|=n} \exp\Big(S_\omega(\phi) - P(\phi)n\Big)\cdot(f\circ\phi_\omega)\, dm_\phi = \int_X f\, dm_\phi \tag{4.9}$$

for all $n \geq 0$ and all continuous functions $f: X \mapsto \mathbb{R}$. Since this equality extends to all bounded measurable functions f, we get

$$\begin{aligned} m_\phi(A) &= \sum_{\tau\in\mathbb{N}^n} \int \exp\Big(S_\tau(\phi) - P(\phi)n\Big)\cdot 1_{\phi_\omega(A)}\circ\phi_\tau\, dm_\phi \\ &\geq \int_A \exp\Big(S_\omega(\phi) - P(\phi)n\Big)\, dm_\phi \end{aligned} \tag{4.10}$$

for all $n \geq 0$, all $\omega \in \mathbb{N}^n$, and all Borel sets $A \subset X$. Now, for each $n \geq 1$ set $X_n = \bigcup_{|\omega|=n} \phi_\omega(X)$. Then $1_{X_n}\circ\phi_\omega = 1$ for all $\omega \in \mathbb{N}^n$. Thus apllying (4.9) to the function $f = 1_{X_n}$ and later to the function $f = 1$, we obtain

$$\begin{aligned} m_\phi(X_n) &= \int_X \sum_{|\omega|=n} \exp\Big(S_\omega(\phi) - P(\phi)n\Big)\cdot(1_{X_n}\circ\phi_\omega)\, dm_\phi \\ &= \int_X \sum_{|\omega|=n} \exp\Big(S_\omega(\phi) - P(\phi)n\Big)\, dm_\phi = \int 1\, dm_\phi = 1. \end{aligned}$$

Hence $m_\phi(J) = m_\phi\Big(\bigcap_{n\geq 1} X_n\Big) = 1$. The proof is complete. ■

Theorem 4.7 *For all $n \geq 1$*

$$Q^{-1} \leq \tilde{L}_0^n(1) \leq Q.$$

Proof Given $n \geq 1$ by (4.9) there exits $x_n \in X$ such that $\tilde{L}_0^n(1)(x_n) \leq 1$. It then follows from Lemma 4.3 that for every $x \in X$, $\tilde{L}_0^n(1)(x) \leq Q$. Similarly by (4.9) there exists $y_n \in X$ such that $\tilde{L}_0^n(1)(y_n) \geq 1$. It then follows from Lemma 4.3 that for every $x \in X$, $\tilde{L}_0^n(1)(x) \geq Q^{-1}$. The proof is finished. ■

So far we have worked downstairs in the compact space X. It is now time to lift our considerations up to the shift space Σ.

Lemma 4.8 *There exists a unique Borel probability measure $\tilde{m}_\phi$ on $\mathbb{N}^\infty$ such that $\tilde{m}_\phi([\omega]) = \int \exp\Big(S_\omega(\phi) - P(\phi)n\Big)\, dm_\phi$ for all $\omega \in \mathbb{N}^*$.*

Proof In view of (4.10) $\sum_{|\omega|=n} \int \exp\Big(S_\omega(\phi) - P(\phi)n\Big)\, dm_\phi = 1$ for all $n \geq 1$ and therefore one can define a Borel probability measure m_n on C_n, the algebra generated by the cylinder sets of the form $[\omega]$, $\omega \in \mathbb{N}^n$, putting $m_n([\omega]) = \int \exp\Big(S_\omega(\phi) - P(\phi)n\Big)\, dm_\phi$. Hence, applying (4.10) again we get for all $\omega \in \mathbb{N}^n$.

$$\begin{aligned}
m_{n+1}([\omega]) &= \sum_{i\in\mathbb{N}} m_{n+1}([\omega i]) = \sum_{i\in\mathbb{N}} \int \exp\Big(S_{\omega i}(\phi) - P(\phi)(n+1)\Big)\, dm_\phi \\
&= \int \sum_{i\in\mathbb{N}} \exp\Big(\sum_{j=1}^{n} \phi^{(\omega_j)} \circ \phi_{\sigma^j(\omega i)} - P(\phi)n + \phi^{(i)} - P(\phi)\Big)\, dm_\phi \\
&= \int \sum_{i\in\mathbb{N}} \exp\Big(S_\omega \circ \phi_i - P(\phi)n\Big) \exp\Big(\phi^{(i)} - P(\phi)\Big)\, dm_\phi \\
&= \int \tilde{L}_0\Big(\exp(S_\omega(\phi) - P(\phi))\Big)\, dm_\phi \\
&= \int \exp\Big(S_\omega(\phi) - P(\phi)\Big)\, dm_\phi = m_n([\omega])
\end{aligned}$$

and therefore in view of Kolmogorov's extension theorem there exists a unique probability measure $\tilde{m}_\phi$ on $\mathbb{N}^\infty$ such that $\tilde{m}_\phi([\omega]) = \tilde{m}_{|\omega|}([\omega])$ for all $\omega \in \mathbb{N}^*$. The proof is finished. ■

As an immediate consequence of this lemma we see that if R is a collection of incomparable words such that $\bigcup_{\omega\in R}[\omega] = \mathbb{N}^\infty$, then we have

$$1 \leq \sum_{\omega\in R} ||\exp\Big(S_\omega(\phi) - P(\phi)|\omega|\Big)||_0 \leq Q \text{ and } Q^{-1} \leq \sum_{\omega\in R} \inf_X \exp\Big(S_\omega(\phi) - P(\phi)|\omega|\Big) \leq 1.$$

Now we are ready to prove that the function ϕ is positive recurrent. Let us first notice that

$$\begin{aligned} L_\phi(1)(\omega) &= \sum_{\tau\in\sigma^{-1}(\omega)} e^{\phi(\tau)} = \sum_{\tau\in\sigma^{-1}(\omega)} \exp\Big(\phi^{(\tau_1)}(\pi(\sigma(\tau)))\Big) \\ &= \sum_{\tau\in\sigma^{-1}(\omega)} \exp\Big(\phi^{(\tau_1)}(\pi(\omega))\Big) = \sum_{i\in\mathbb{N}} e^{\phi^{(i)}(\pi(\omega))} = \tilde{L}_\phi(1)(\pi(\omega)). \end{aligned}$$

Since $\tilde{L}_0 = e^{-P(\phi)}\tilde{L}_\phi$, it then follows from Theorem 4.7 that as M we can take Q. In order to demonstrate that the function ϕ is positive recurrent we first show that

$$\frac{Z_n(\phi, i)}{L_\phi^n(1)(\omega)} \leq Q$$

for all $n \geq 1$, $\omega \in \Sigma$, and some constant $M_i > 0$. So fix $\omega \in \Sigma$. We shall define an injection j from $\{\rho \in \Sigma : \sigma^n(\rho) = \rho \text{ and } \rho_1 = i\}$ into $\sigma^{-n}(\omega)$ as follows: $j(\rho) = \rho_1\rho_2\ldots\rho_n\omega$. Now, by Lemma 4.3

$$\left|\sum_{j=0}^{n-1} \phi(\sigma^j(\rho)) - \sum_{j=0}^{n-1} \phi(\sigma^j(j(\rho)))\right| \leq \log Q$$

and therefore $Z_n(\phi, i) \leq QL_\phi^n(1)(\omega)$. Thus by Theorem 4.7 and the definition of the operators $\tilde{L}_0$ and L_0, $Z_n(\phi, i) \leq M_i\lambda^n$, where $M_i = Q^2$.

Now we shall prove that $Z_n(\phi, i) \geq M_i'\lambda^n$ for some constant M_i' and all $n \geq 1$. We demonstrate first that for all $n \geq 1$ and all $i \in \mathbb{N}$

$$L_0(1_{[i]}) \geq \tilde{m}_\phi([i]).$$

Indeed, since $\int L_0(1_{[i]})\, d\tilde{m}_\phi = \int 1_{[i]}\, d\tilde{m}_\phi = \tilde{m}_\phi([i]) > 0$, there exists $\tau \in \Sigma$ such that $L_0(1_{[i]})(\tau) \geq \tilde{m}_\phi([i])$. It the follows from Lemma 4.3 that for every $\omega \in \Sigma$

$$\begin{aligned} L_0^n(1_{[i]})(\omega) &= \sum_{\rho \in \sigma^{-n}(\omega)} \exp\Big(\sum_{j=0}^{n-1} \phi \circ \sigma^j(\rho) 1_{[i]}(\rho)\Big) \\ &\geq Q^{-1} \sum_{\rho \in \sigma^{-n}(\tau)} \exp\Big(\sum_{j=0}^{n-1} \phi \circ \sigma^j(\rho) 1_{[i]}(\rho)\Big) = Q^{-1} L_0^n(1_{[i]})(\tau) \\ &\geq \tilde{m}_\phi([i]). \end{aligned}$$

Hence $L_\phi^n(1_{[i]})(\omega) \geq \lambda^n \tilde{m}_\phi([i])$.

So, in order to conclude the proof that ϕ is positively recurrent it suffices now to show that

$$\frac{Z_n(\phi, i)}{L_\phi^n(1_{[i]})(\omega)} \geq M_i''$$

for all $n \geq 1$, all $\omega \in \Sigma$ and some constant $M_i'' > 0$. Indeed, we shall define an injection k from $\sigma^{-n}(\omega) \cap [i]$ to $\{\rho \in \Sigma : \sigma^n(\rho) = \rho \text{ and } \rho_1 = i\}$ by taking as $k(\tau)$ the infinite concatenation of the first n words of τ. Then by Lemma 4.3,

$$\left| \sum_{j=0}^{n-1} \phi(\sigma^j(\tau)) - \sum_{j=0}^{n-1} \phi(\sigma^j(k(\tau))) \right| \leq \log Q$$

and therefore

$$L_\phi^n(1_{[i]})(\omega) = \sum_{\rho \in \sigma^{-n}(\omega)} \exp\Big(\sum_{j=0}^{n-1} \phi \circ \sigma^j(\rho) 1_{[i]}(\rho)\Big)$$

$$
\begin{aligned}
&= \sum_{\rho\in\sigma^{-n}(\omega)\cap[i]} \exp\Big(\sum_{j=0}^{n-1}\phi\circ\sigma^j(\rho)\Big)\\
&\le \sum_{\rho\in\sigma^{-n}(\omega)\cap[i]} \exp\Big(\sum_{j=0}^{n-1}\phi\circ\sigma^j(k(\rho)) + \log Q\Big)\\
&\le Q\sum \exp\Big(\sum_{j=0}^{n-1}\phi\circ\sigma^j(\rho)\Big) = QZ_n(\phi,i),
\end{aligned}
$$

where the last summation is taken over all elements ω satisfying $\sigma^n(\omega)=\omega$ and $\omega_1 = i$. So, the proof of the positive recurrence of ϕ is complete taking Q^{-1} as M_i'' and $Q^{-1}\tilde{m}_\phi([i])$ as M_i'.

Now we pass to proving the existence of the measure ν and the function h. We begin with the following two facts.

Lemma 4.9 *The measures m_ϕ and $\tilde{m}_\phi\circ\pi^{-1}$ are equal.*

Proof Let $A\subset J$ be an arbitrary closed subset of J and for every $n\ge 1$ let $A_n = \{\omega\in\mathbb{N}^n : \phi_\omega(X)\cap A\ne\emptyset\}$. In view of (4.9) applied to the characteristic function 1_A we have for all $n\ge 1$

$$
\begin{aligned}
m_\phi(A) &= \sum_{\omega\in\mathbb{N}^n}\int \exp\Big(S_\omega(\phi)-P(\phi)|\omega|\Big)(1_A\circ\phi_\omega)\,dm_\phi\\
&= \sum_{\omega\in A_n}\int \exp\Big(S_\omega(\phi)-P(\phi)|\omega|\Big)(1_A\circ\phi_\omega)\,dm_\phi\\
&\le \sum_{\omega\in A_n}\int \exp\Big(S_\omega(\phi)-P(\phi)|\omega|\Big)\,dm_\phi = \sum_{\omega\in A_n}\tilde{m}_\phi([\omega]) = \tilde{m}_\phi\Big(\bigcup_{\omega\in A_n}[\omega]\Big)
\end{aligned}
$$

Since the family of sets $\{\bigcup_{\omega\in A_n}[\omega] : n\ge 1\}$ is descending and $\bigcap_{n\ge 1}\bigcup_{\omega\in A_n}[\omega] = \pi^{-1}(A)$ we therefore get $m_\phi(A)\le \lim_{n\to\infty}\tilde{m}_\phi\Big(\bigcup_{\omega\in A_n}[\omega]\Big) = \tilde{m}_\phi(\pi^{-1}(A))$. Since the limit set J is a metric space , using the Baire classification of Borel sets we easily see that this inequality extends to the family of all Borel subsets of J. Since both measures m_ϕ and $\tilde{m}_\phi \circ \pi^{-1}$ are probabilistic we get $m_\phi = \tilde{m}_\phi \circ \pi^{-1}$. ■

We recall that an invariant mesure of a metric dynamical system is said to be *totally ergodic* if it is ergodic with respect to all the iterates of the system under consideration.

Theorem 4.10 *There exists a unique totally ergodic σ-invariant probability measure $\tilde{\mu}_\phi$ absolutely continuous with respect to $\tilde{m}_\phi$. Moreover $\tilde{\mu}_\phi$ is equivalent with $\tilde{m}_\phi$ and the Radon-Nikodym derivative is bounded from above and from below, that is $Q^{-1} \leq d\tilde{\mu}_\phi / d\tilde{m}_\phi \leq Q$.*

Proof First notice that, using (4.9) and Lemma 4.3, for each $\omega \in \mathbb{N}^*$ and each $n \geq 0$ we have

$$\begin{aligned}
\tilde{m}_\phi(\sigma^{-n}([\omega])) &= \sum_{\tau \in \mathbb{N}^n} \tilde{m}_\phi([\tau\omega]) = \sum_{\tau \in \mathbb{N}^n} \int \exp\Big(S_{\tau\omega}(\phi) - P(\phi)|\tau\omega|\Big)\, dm_\phi \\
&\geq \sum_{\tau \in \mathbb{N}^n} Q^{-1} || \exp\Big(S_\tau(\phi) - P(\phi)|\tau|\Big)||_0 \int \exp\Big(S_\omega(\phi) - P(\phi)|\omega|\Big)\, dm_\phi \\
&= Q^{-1} \int \exp\Big(S_\omega(\phi) - P(\phi)|\omega|\Big)\, dm_\phi \sum_{\tau \in \mathbb{N}^n} || \exp\Big(S_\tau(\phi) - P(\phi)|\tau|\Big)||_0 \\
&\geq Q^{-1} \tilde{m}_\phi([\omega]) \tilde{m}_\phi(\mathbb{N}^\infty) = Q^{-1} \tilde{m}_\phi([\omega])
\end{aligned}$$

and

$$\begin{aligned}
\tilde{m}_\phi(\sigma^{-n}([\omega])) &= \sum_{\tau \in \mathbb{N}^n} \tilde{m}_\phi([\tau\omega]) = \sum_{\tau \in \mathbb{N}^n} \int \exp\Big(S_{\tau\omega}(\phi) - P(\phi)|\tau\omega|\Big)\, dm_\phi \\
&\leq \sum_{\tau \in \mathbb{N}^n} || \exp\Big(S_\tau(\phi) - P(\phi)|\tau|\Big)||_0 \int \exp\Big(S_\omega(\phi) - P(\phi)|\omega|\Big)\, dm_\phi \\
&= \int \exp\Big(S_\omega(\phi) - P(\phi)|\omega|\Big)\, dm_\phi \sum_{\tau \in \mathbb{N}^n} || \exp\Big(S_\tau(\phi) - P(\phi)|\tau|\Big)||_0 \\
&\leq Q \tilde{m}_\phi([\omega]).
\end{aligned}$$

Let now L be a Banach limit defined on the Banach space of all bounded sequences of real numbers. We define $\mu([\omega]) = \mathrm{L}\Big((\tilde{m}_\phi(\sigma^{-n}([\omega])))_{n\ge 0}\Big)$. Hence $Q^{-1}\tilde{m}_\phi([\omega]) \le \mu([\omega]) \le Q\tilde{m}_\phi([\omega])$ and therefore it is not difficult to check that the formula $\mu(A) = \mathrm{L}\Big((\tilde{m}_\phi(\sigma^{-n}(A)))_{n\ge 0}\Big)$ defines a finite non-zero finitely additive measure on Borel sets of $\mathbb{N}^\infty$ satisfying $Q^{-1}\tilde{m}_\phi(A) \le \mu(A) \le Q\tilde{m}_\phi(A)$. An application of Kolmogorov's extension theorem shows that the function μ extends to a σ-invariant Borel probability (σ-additive) measure $\tilde{\mu}_\phi$ on $\mathbb{N}^\infty$ satisfying the formula

$$Q^{-1}\tilde{m}_\phi(A) \le \tilde{\mu}_\phi(A) \le Q\tilde{m}_\phi(A)$$

for every Borel set $A \subset \mathbb{N}^\infty$ with, perhaps, a larger constant Q. Thus, to complete the proof of our theorem we only need to show total ergodicity of $\tilde{\mu}_\phi$ or equivalently of $\tilde{m}_\phi$. Toward this end take a Borel set $A \in \mathbb{N}^\infty$ with $\tilde{m}_\phi(A) > 0$. Since the nested family of sets $\{[\tau] : \tau \in \mathbb{N}^*\}$ generates the Borel σ-algebra on $\mathbb{N}^\infty$, for every $n \ge 0$ and every $\omega \in \mathbb{N}^n$ we can find a subfamily Z of $\mathbb{N}^*$ consisting of mutually incomparable words and such that $A \subset \bigcup\{[\tau] : \tau \in Z\}$ and $\sum_{\tau\in Z}\tilde{m}_\phi([\omega\tau]) \le 2\tilde{m}_\phi(\omega A)$, where $\omega A = \{\omega\rho : \rho \in A\}$. Then

$$\begin{aligned}
\tilde{m}_\phi\Big(\sigma^{-n}(A)\cap[\omega]\Big) &= \tilde{m}_\phi(\omega A) \ge \frac{1}{2}\sum_{\tau\in Z}\tilde{m}_\phi([\omega\tau]) \\
&= \frac{1}{2}\sum_{\tau\in Z}\int \exp\Big(S_{\omega\tau}(\phi) - P(\phi)|\omega\tau|\Big)\,dm_\phi \\
&\ge \frac{1}{2Q}\exp\Big(S_\omega(\phi) - P(\phi)|\omega|\Big)||_0 \\
&\quad \sum_{\tau\in Z}\int \exp\Big(S_\tau(\phi) - P(\phi)|\tau|\Big)\,dm_\phi \\
&\ge \frac{1}{2Q}\int \exp\Big(S_\omega(\phi) - P(\phi)|\omega|\Big)\,dm_\phi \sum_{\tau\in Z}\tilde{m}_\phi([\tau])
\end{aligned}$$

$$\geq \frac{1}{2Q}\tilde{m}_\phi([\omega])\tilde{m}_\phi\Big(\bigcup\{[\tau] : \tau \in Z\}\Big) \geq \frac{1}{2Q}\tilde{m}_\phi(A)\tilde{m}_\phi([\omega]).$$

Therefore

$$\begin{aligned}\tilde{m}_\phi\Big(\sigma^{-n}(\mathbb{N}^\infty \setminus A) \cap [\omega]\Big) &= \tilde{m}_\phi\Big([\omega] \setminus \sigma^{-n}(A) \cap [\omega]\Big) = \tilde{m}_\phi([\omega]) - \tilde{m}_\phi\Big(\sigma^{-n}(A) \cap [\omega]\Big) \\ &\leq \Big(1 - (2Q)^{-1}\tilde{m}_\phi(A)\Big)\tilde{m}_\phi([\omega]).\end{aligned}$$

Hence for every Borel set $A \subset \mathbb{N}^\infty$ with $\tilde{m}_\phi(A) < 1$, for every $n \geq 0$, and for every $\omega \in \mathbb{N}^n$ we get

$$\tilde{m}_\phi\Big(\sigma^{-n}(A) \cap [\omega]\Big) \leq \Big(1 - (2Q)^{-1}(1 - \tilde{m}_\phi(A))\Big)\tilde{m}_\phi([\omega]). \tag{4.11}$$

In order to conclude the proof of total ergodicity of σ suppose that $\sigma^{-r}(A) = A$ for some integer $r \geq 1$ and some Borel set A with $0 < \tilde{m}_\phi(A) < 1$. Put $\gamma = 1 - (2Q)^{-1}(1 - \tilde{m}_\phi(A))$. Note that $0 < \gamma < 1$. In view of (4.11), for every $\omega \in (\mathbb{N}^r)^*$ we get $\tilde{m}_\phi(A \cap [\omega]) = \tilde{m}_\phi\Big(\sigma^{-|\omega|}(A) \cap [\omega]\Big) \leq \gamma\tilde{m}_\phi([\omega])$. Take now $\eta > 1$ so small that $\gamma\eta < 1$ and choose a subfamily R of $(\mathbb{N}^r)^*$ consisting of mutually incomparable words and such that $A \subset \bigcup\{[\omega] : \omega \in R\}$ and $\tilde{m}_\phi\Big(\bigcup\{[\omega] : \omega \in R\}\Big) \leq \eta\tilde{m}_\phi(A)$. Then $\tilde{m}_\phi(A) \leq \sum_{\omega\in R}\tilde{m}_\phi(A \cap [\omega]) \leq \sum_{\omega\in R}\gamma\tilde{m}_\phi([\omega]) = \gamma\tilde{m}_\phi\Big(\bigcup\{[\omega] : \omega \in R\}\Big) \leq \gamma\eta\tilde{m}_\phi(A) < \tilde{m}_\phi(A)$. This contradiction finishes the proof. ■

Set $\nu = \tilde{m}_\phi$. Clearly our assumption $\phi_i(X) \cap \phi_j(X) = \emptyset$ for $i, j \in \mathbb{N}$, $i \neq j$ implies that $\pi : \Sigma \mapsto J$ is a homeomorphism; in particuluar, in view of Lemma 4.8, it establishes a measure preserving isomorphism between measure spaces (Σ, ν) and (J, m_ϕ). To check that $L_\phi^*(\nu) = \lambda\nu$ take $g \in C_b(\Sigma)$ and compute

$$\int g\,dL_0^*(\nu) = \int L_0(g)\,d\nu = \int L_0(g)(\pi^{-1}(x))\,d\nu \circ \pi^{-1}(x) = \int L_0(g)(\pi^{-1}(x))\,dm_\phi$$

$$
\begin{aligned}
&= \int \sum_{\tau \in \sigma^{-1}(\pi^{-1}(x))} \exp(\phi(\tau) - P(\phi)) g(\tau)\, dm_\phi \\
&= \int \sum_{i \in \mathbb{N}} \exp\left(\phi^{(i)}(x) - P(\phi)\right) g \circ \pi^{-1}(\phi_i(x))\, dm_\phi(x) \\
&= \int \tilde{L}_0(g \circ \pi^{-1})\, dm_\phi = \int g \circ \pi^{-1}\, dm_\phi = \int g\, d\nu.
\end{aligned}
$$

Thus $L_0(\nu) = \nu$ and by the definition of L_0 and L_0^*, $L_\phi^*(\nu) = \lambda\nu$. The fact that $L_\phi(h) = \lambda h$ follows immediately from the definition of the operator L_0 and Theorem 4.10, where $h = d\tilde{\mu}_\phi / d\tilde{m}_\phi$. Theorem 4.10 also implies that h is bounded away from zero and infinity. In order to obtain Hölder continuity of the function h and two convergence statements claimed in Theorem 4.2 one may argue as follows: A well-known computation (see [DU], comp [MU1]) shows that L_0 acts on the Banach space of bounded uniformly continuous functions on $\mathbb{N}^\infty$ as an almost periodic operator (see [Ly], comp. [DU] and [MU1]). Using Theorem 4.10 and the theory of positive operators on lattices (see [Sc]) one then proves as in [DU] that 1 is the only spectral point of modulus 1 and additionally that 1 is a simple eigenvalue of L_0. These facts and almost periodicity imply the first convergence statement of Theorem 4.2 and uniform continuity of h. A similar computation as above produces constants $0 < \gamma < 1$, $n \geq 1$ and $C \geq 0$ such that

$$
||L_0^n(\gamma)||_\beta \leq C||\gamma||_0 + \gamma||g||_\beta,
$$

where $||\gamma||_\beta = V_\beta(\gamma) + ||g||_0$. This is the Ionescu-Tulcea and Marinescu inequality. Using this inequality and Theorem 4.6 one checks that the assumptions of the theorem of Ionescu-Tulcea and Marinescu (see [IM], comp. [PU]) are satisfied. This theorem gives a nice spectral decomposition of the operator L_0 acting on the space H_β of bounded Hölder continuous functions of order β. Having this, a relatively straightforward reasoning (comp. [PU]) shows Hölder continuity of h and the second convergence statement of Theorem 4.2.

CHAPTER 5

EQUILIBRIUM STATES

In this chapter we further investigate the measure $\tilde{\mu}_\phi$ introduced in Theorem 4.10 and invariant with respect to the shift map σ. Following the approach from [KH] or [Wa] we introduce the notation for a measure-theoretic entropy. Let $(X, \mathcal{B}, m)$ be a probability space and I a finite or countable set of indices.

The *entropy* of a measurable partition ξ is given by

$$H_m(\xi) = -\sum_{\alpha \in I} m(A_\alpha) \log m(A_\alpha),$$

where $\xi = \{A_\alpha \in \mathcal{B} : \ \alpha \in I\}$. Then for a measure-preserving transformation T we define the *metric entropy* of the transformation T relative to the partition ξ by

$$h_m(T, \xi) = \lim_{n \to \infty} \frac{1}{n} H\Big(\xi \vee T^{-1}(\xi) \vee \ldots \vee T^{-(n-1)}(\xi)\Big).$$

It is known that the limit in the above formula always exists. Finally, the *entropy of T with respect to m* is

$$h_m(T) = \sup\{h_m(T, \xi) : \ \xi \text{ is a measurable partition with } H_m(\xi) < \infty\}.$$

We begin with the following technical result.

Lemma 5.1 *The following 3 conditions are equivalent*

(a) $\int -\phi \, d\tilde{\mu}_\phi < \infty$.

(b) $\sum_{i\in\mathbb{N}} \inf(-\phi|_{[i]}) \exp(\inf \phi|_{[i]}) < \infty$.

(c) $H_{\tilde{\mu}_\phi}(\alpha) = \sum_{i\in\mathbb{N}} -\tilde{\mu}_\phi([i]) \log \tilde{\mu}_\phi([i]) < \infty$*, where* $\alpha = \{[i] : i \in \mathbb{N}\}$ *is the partition of* Σ *into initial cylinders of length 1.*

Proof $(a) \Rightarrow (b)$. Suppose that $\int -\phi \, d\tilde{\mu}_\phi < \infty$. It means that $\sum_{i\in\mathbb{N}} \int_{[i]} -\phi \, d\tilde{\mu}_\phi < \infty$ and consequently

$$\begin{aligned}
\infty &> \sum_{i\in\mathbb{N}} \inf(-\phi|_{[i]}) \int_{[i]} d\tilde{\mu}_\phi = \sum_{i\in\mathbb{N}} \inf(-\phi|_{[i]}) \int_{[i]} h \, d\tilde{m}_\phi \\
&\geq Q^{-1} \sum_{i\in\mathbb{N}} \inf(-\phi|_{[i]}) \tilde{m}_\phi([i]) \\
&= Q^{-1} \sum_{i\in\mathbb{N}} \inf(-\phi|_{[i]}) \int_X \exp(\phi^{(i)}(x) - P(\phi)) \, dm_\phi(x) \\
&= Q^{-1} e^{-P(\phi)} \sum_{i\in\mathbb{N}} \inf(-\phi|_{[i]}) \int_X \exp(\phi^{(i)}(x)) \, dm_\phi(x).
\end{aligned}$$

Thus,

$$\begin{aligned}
\infty &> \sum_{i\in\mathbb{N}} \inf(-\phi|_{[i]}) \int_X \exp(\phi^{(i)}(x)) \, dm_\phi(x) \geq \sum_{i\in\mathbb{N}} \inf(-\phi|_{[i]}) \exp(\inf_X(\phi^{(i)}) \\
&= \sum_{i\in\mathbb{N}} \inf(-\phi|_{[i]}) \exp(\inf \phi|_{[i]}).
\end{aligned}$$

$(b) \Rightarrow (c)$. Now suppose that $\sum_{i\in\mathbb{N}} \inf(-\phi|_{[i]}) \exp(\inf \phi|_{[i]}) < \infty$. We shall show that $H_{\tilde{\mu}_\phi}(\alpha) < \infty$. By definition,

$$H_{\tilde{\mu}_\phi}(\alpha) = \sum_{i\in\mathbb{N}} -\tilde{\mu}_\phi([i]) \log \tilde{\mu}_\phi([i]) \leq \sum_{i\in\mathbb{N}} -Q\tilde{m}_\phi([i])(\log \tilde{m}_\phi([i]) - \log Q).$$

But, $\sum_{i\in\mathbb{N}} -Q\tilde{m}_\phi([i])(-\log Q) = Q \log Q$, so it suffices to show that

$$\sum_{i\in\mathbb{N}} -\tilde{m}_\phi([i]) \log \tilde{m}_\phi([i]) < \infty.$$

However,

$$\begin{aligned}\sum_{i\in\mathbb{N}} -\tilde{m}_\phi([i]) \log \tilde{m}_\phi([i]) &= \sum_{i\in\mathbb{N}} -\tilde{m}_\phi([i]) \log \Big(\int_X \exp(\phi^{(i)} - P(\phi)) \Big) \, dm_\phi \\ &\leq \sum_{i\in\mathbb{N}} -\tilde{m}_\phi([i]) (\inf_X \phi^{(i)} - P(\phi)).\end{aligned}$$

Since $\sum_{i\in\mathbb{N}} \tilde{m}_\phi([i])P(\phi) = P(\phi)$, it suffices to show that $\sum_{i\in\mathbb{N}} -\tilde{m}_\phi([i]) \inf_X \phi^{(i)} < \infty$. And indeed, using Lemma 4.3 we get

$$\sum_{i\in\mathbb{N}} -\tilde{m}_\phi([i]) \inf_X \phi^{(i)} = \sum_{i\in\mathbb{N}} \tilde{m}_\phi([i]) \sup_X(-\phi^{(i)}) \leq \sum_{i\in\mathbb{N}} \tilde{m}_\phi([i]) (\inf_X(-\phi^{(i)}) + \log Q).$$

Since $\sum_{i\in\mathbb{N}} \tilde{m}_\phi([i]) \log Q = \log Q$, it is enough to show that

$$\sum_{i\in\mathbb{N}} \tilde{m}_\phi([i]) \inf_X(-\phi^{(i)}) < \infty.$$

In fact,

$$\sum_{i\in\mathbb{N}} \tilde{m}_\phi([i]) \inf_X(-\phi^{(i)}) = \sum_{i\in\mathbb{N}} \int \exp(\phi^{(i)} - P(\phi)) \, dm_\phi \inf_X(-\phi^{(i)}).$$

But in view of (4.5) $\phi^{(i)}$ are negative everywhere for all i large enough, say $i \geq k$. Then using Lemma 4.3 again we get

$$\sum_{i\geq k} \tilde{m}_\phi([i]) \inf_X(-\phi^{(i)}) \leq e^{-P(\phi)} Q \sum_{i\geq k} \exp(\inf_X(\phi^{(i)})) \inf_X(-\phi^{(i)})$$

which is finite due to our assumption. Hence, $\sum_{i\in\mathbb{N}} \tilde{m}_\phi([i]) \inf_X(-\phi^{(i)}) < \infty$.

$(c) \Rightarrow (a)$. Suppose that $H_{\tilde{\mu}_\phi}(\alpha) < \infty$. We need to show that $\int -\phi \, d\tilde{\mu}_\phi < \infty$. We have

$$\infty > H_{\tilde{\mu}_\phi}(\alpha) = \sum_{i\in\mathbb{N}} -\tilde{m}_\phi([i]) \log(\tilde{m}_\phi([i])) \geq \sum_{i\in\mathbb{N}} -\tilde{m}_\phi([i]) (\inf(\phi|_{[i]} - P(\phi) + \log Q)).$$

Hence $\sum_{i\in\mathbb{N}} -\tilde{m}_\phi([i])\inf(\phi|_{[i]}) < \infty$ and therefore

$$\int -\phi\, d\tilde{\mu}_\phi = \sum_{i\in\mathbb{N}} \int_{[i]} -\phi\, d\tilde{\mu}_\phi \le \sum_{i\in\mathbb{N}} \sup(-\phi|_{[i]})\tilde{m}_\phi([i]) = \sum_{i\in\mathbb{N}} -\inf(\phi|_{[i]})\tilde{m}_\phi([i]) < \infty.$$

The proof is complete. ■

By Theorem 3 of [Sa] we know that $\sup\{h_\mu(\sigma) + \int \phi\, d\mu\} = P(\phi)$, where the supremum is taken over all σ-invariant probability measures such that $\int -\phi\, d\mu < \infty$. We call a σ-invariant probability measure μ an *equilibrium state* of the potential ϕ if $h_\mu(\sigma) + \int \phi\, d\mu = P(\phi)$.

Given $\omega \in \mathbb{N}^*$, say $\omega \in \mathbb{N}^n$ by $\sigma_\omega^{-n} : \Sigma \to \Sigma$ we denote the map defined by the formula

$$\sigma_\omega^{-n}(\tau) = \omega\tau.$$

Notice that σ_ω^{-n} is a continuous branch of σ^{-n}. Given a Borel probability shift-invariant measure μ on Σ we call the function $J_\mu : \Sigma \to (0, +\infty)$ the Jacobian of the shift map σ with respect to the measure μ if for every Borel set $A \subset \Sigma$

$$\mu(\sigma_i^{-1}(A)) = \int_A \frac{1}{J_\mu \circ \sigma_i^{-1}}\, d\mu.$$

By $L_\mu : L^\infty \to L^\infty$ we denote the Perron-Frobenius operator of the measure μ, i.e. the operator defined by the formula

$$L_\mu(g)(\omega) = \sum_{i\in\mathbb{N}} J_\mu(i\omega)g(i\omega).$$

We shall prove the following.

Lemma 5.2 *If μ is an equilibrium state for the shift map $\sigma : \Sigma \to \Sigma$ and potential ϕ such that $\int -\phi \, d\mu < \infty$, then*

$$J_\mu = \frac{\rho \circ \sigma}{\rho} \cdot \exp(P(\phi) - \phi)$$

μ almost everywhere, where $\rho = d\tilde{\mu}_\phi / d\tilde{m}_\phi$ is the Radon-Nikodym derivative introduced in Theorem 4.10.

Proof Let $L : C_b(\Sigma) \to C_b(\Sigma)$ be the Perron-Frobenius operator defined by formula

$$L(g)(\omega) = \sum_{\tau \in \sigma^{-1}(\omega)} \exp(\phi(\tau) - P(\phi))\phi(\tau) = \sum_{i \in \mathbb{N}} \exp(\phi(i\omega) - P(\phi))\phi(i\omega).$$

The density $\rho = d\tilde{\mu}_\phi / d\tilde{m}_\phi$ existing due to Theorem 4.10 is its fixed point and according to Theorem 5.2 of [U] ρ has a version in $C_b(\Sigma)$, even Hölder continuous. Therefore, using inequality $x \geq 1 + \log x$ we can write

$$\begin{aligned}
1 &= \int 1 \, d\mu = \int \frac{L(\rho)}{\rho} \, d\mu = \int L_\mu \Big(\frac{\rho \cdot \exp(\phi - P(\phi))}{J_\mu^{-1} \cdot \rho \circ \sigma} \Big) \, d\mu \\
&= \int \frac{\rho \cdot \exp(\phi - P(\phi))}{J_\mu^{-1} \cdot \rho \circ \sigma} \, d\mu \geq 1 + \int \log \Big(\frac{\rho \cdot \exp(\phi - P(\phi))}{J_\mu^{-1} \cdot \rho \circ \sigma} \Big) \, d\mu \\
&= 1 + \int \log \rho \, d\mu - \int \log(\rho \circ \sigma) \, d\mu + \int (\phi - P(\phi)) \, d\mu + \int \log J_\mu \, d\mu \\
&= 1 + \int \phi \, d\mu - P(\phi) + h_\mu(\sigma) = 1.
\end{aligned}$$

Notice that we were in position to write the inequality sign and the equality sign following it since by our assumptions $\int \phi \, d\mu$ is finite and since $\log J_\mu$ is a non-negative function. Since $x = 1 + \log x$ if and only if $x = 1$, we conclude from this display that $\frac{\rho \cdot \exp(\phi - P(\phi))}{J_\mu^{-1} \cdot \rho \circ \sigma} = 1$ μ-a.e. The proof is complete. ■

Theorem 5.3 *If $\sum_{i\in\mathbb{N}} \inf(-\phi|_{[i]}) \exp(\inf \phi|_{[i]}) < \infty$, then $\tilde{\mu}_\phi$ is a unique equilibrium state of the potential ϕ satisfying $\int -\phi \, d\tilde{\mu}_\phi < \infty$.*

Proof It follows from Lemma 5.1 that $\int -\phi \, d\tilde{\mu}_\phi < \infty$. To show that $\tilde{\mu}_\phi$ is an equilibrium state of the potential ϕ consider $\alpha = \{[i] : i \in \mathbb{N}\}$, the partition of Σ into initial cylinders of length one. By Lemma 5.1, $H_{\tilde{\mu}_\phi}(\alpha) < \infty$. Applying the Shannon-McMillan-Breiman theorem and the Birkhoff ergodic theorem we therefore get for $\tilde{\mu}_\phi$-a.e. $\omega \in \Sigma$

$$\begin{aligned}
h_{\mu_\phi}(\sigma) &\geq h_{\mu_\phi}(\sigma, \alpha) = \lim_{n\to\infty} \frac{-1}{n} \log([\omega|_n]) \\
&= \lim_{n\to\infty} \frac{-1}{n} \log \Big(\int \exp(S_\omega(\phi)(x) \, d\mu_\phi - P(\phi)n) \Big) \\
&= \lim_{n\to\infty} \frac{-1}{n} \log \Big(\int \exp \Big(\sum_{j=0}^{n-1} \phi(\sigma^j(\omega|_n \tau)) \, d\mu_\phi(\tau) - P(\phi)n \Big) \Big) \\
&\geq \limsup_{n\to\infty} \frac{-1}{n} \log \Big(\int \exp \Big(\sum_{j=0}^{n-1} \phi(\sigma^j(\omega)) + \log Q - P(\phi)n \Big) \Big) \\
&= \lim_{n\to\infty} \frac{-1}{n} \sum_{j=0}^{n-1} \phi(\sigma^j(\omega)) + P(\phi) = -\int \phi \, d\tilde{\mu}_\phi + P(\phi).
\end{aligned}$$

Hence $h_{\mu_\phi}(\sigma) + \int \phi \, d\tilde{\mu}_\phi \geq P(\phi)$, which in view of the variational principle (see Theorem 3 in [Sa]) implies that $\tilde{\mu}_\phi$ is an equilibrium state for the potential ϕ.

We shall now prove that $\tilde{\mu}$ is the only equilibrium state for ϕ. So, suppose that μ is an equilibrium state. Fix $\omega \in \mathbb{N}^*$, say $\omega \in \mathbb{N}^n$. We should notice that Lemma 4.8, Theorem 4.10, and Lemma 4.3 imply that for every $\rho \in \Sigma$

$$Q^{-2} \leq \frac{\tilde{\mu}_\phi([\rho|_n])}{\exp(\sum_{j=0}^{n-1} \phi \circ \sigma^j(\rho) - P(\phi)n)} \leq Q^2. \tag{5.1}$$

Now, it follows from Lemma 5.2, Lemma 4.3 and the formula (5.1) that for every

$\gamma \in [\omega]$ we get

$$\begin{aligned}
\mu([\omega]) &= \mu(\sigma_\omega^{-n}(\Sigma)) = \int J_\mu^{-1}(\omega\tau) J_\mu^{-1}(\sigma(\omega\tau)) \ldots J_\mu^{-1}(\sigma^{n-1}(\omega\tau))\, d\mu(\tau) \\
&= \int \frac{\rho(\omega\tau)}{\rho(\sigma(\omega\tau))} \exp(\phi(\omega\tau) - P(\phi)) \frac{\rho(\sigma(\omega\tau))}{\rho(\sigma^2((\omega\tau))} \exp(\phi(\sigma(\omega\tau)) - P(\phi)) \ldots \\
&\ldots \frac{\rho(\sigma^{n-1}(\omega\tau))}{\rho(\sigma^n((\omega\tau))} \exp(\phi(\sigma^{n-1}(\omega\tau)) - P(\phi)n)\, d\mu(\tau) \\
&= \int \frac{\rho(\omega\tau)}{\rho(\tau)} \exp\Big(\sum_{j=0}^{n-1} \phi \circ \sigma^j(\omega\tau) - P(\phi)n\Big) \\
&\le Q^2 \int \exp\Big(\sum_{j=0}^{n-1} \phi \circ \sigma^j(\omega\tau) - P(\phi)n\Big) \\
&\le Q^3 \exp\Big(\sum_{j=0}^{n-1} \phi \circ \sigma^j(\gamma) - P(\phi)n\Big) \le Q^5 \tilde{\mu}_\phi([\omega]).
\end{aligned}$$

Hence, the invariant measure μ is absolutely continuous with respect to ergodic invariant measure $\tilde{\mu}_\phi$. The proof is finished. ■

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